

THE AVERAGE FIELD APPROXIMATION FOR ALMOST BOSONIC EXTENDED ANYONS

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ABSTRACT. Anyons are 2D or 1D quantum particles with intermediate statistics, interpolating between bosons and fermions. We study the ground state of a large number N of 2D anyons, in a scaling limit where the statistics parameter α is proportional to N^{-1} when $N \rightarrow \infty$. This means that the statistics is seen as a “perturbation from the bosonic end”. We model this situation in the magnetic gauge picture by bosons interacting through long-range magnetic potentials. We assume that these effective statistical gauge potentials are generated by magnetic charges carried by each particle, smeared over discs of radius R (extended anyons). Our method allows to take $R \rightarrow 0$ not too fast at the same time as $N \rightarrow \infty$. In this limit we rigorously justify the so-called “average field approximation”: the particles behave like independent, identically distributed bosons interacting via a self-consistent magnetic field.

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1. INTRODUCTION

In lower dimensions there are possibilities for quantum statistics different from bosons and fermions, so called intermediate or fractional statistics. Due to the prospect that such particles, termed anyons (as in *anything* in between bosons and fermions), could arise as

effective quasiparticles in many-body quantum systems confined to lower dimensions, there has been a great interest over the last three decades in figuring out the behavior of such statistics (see [15, 19, 20, 23, 39, 42, 53] for extensive reviews). In one dimension, one can view the Lieb-Liniger model [28] as providing an example of effective interpolating statistics. Although initially regarded as purely hypothetical while at the same time offering substantial analytical insight thanks to its exact solvability, this system has now been realized concretely in the laboratory [21, 43]. Much less is known concerning fractional statistics in the two-dimensional setting, conjectured [1] to be relevant for the fractional quantum Hall effect (see [16, 22] for review), and it is indeed a very challenging theoretical question to figure out even the ground state properties of an ideal 2D many-anyon gas, parameterized by a single statistics phase $e^{i\pi\alpha}$, or a periodic real parameter α (with $\alpha = 0$ corresponding to bosons and $\alpha = 1$ to fermions). On the rigorous analytical side, some recent progress in this direction has been achieved in [34, 35, 36] where a better understanding of the ground state energy was obtained in the case that α is an odd numerator fraction. Numerous approximative descriptions have also been proposed over the years, such as e.g. in [6] where the problem was approached from both the bosonic and the fermionic ends, with a harmonic trapping potential. Here, equipped with new methods in many-body spectral theory, we will re-visit this question from the perspective of a perturbation around bosons, i.e. in a regime where α is small.

1.1. The model. One may formally think of 2D anyons with statistics parameter α as being described by an N -body wave function of the form

$$\Psi(x_1, \dots, x_N) = \prod_{j < k} e^{i\alpha\phi_{jk}} \tilde{\Psi}(x_1, \dots, x_N), \quad \phi_{jk} = \arg \frac{x_j - x_k}{|x_j - x_k|},$$

where $\tilde{\Psi}$ is a bosonic wave function, i.e. symmetric under particle exchange. This way

$$\Psi(x_1, \dots, x_j, \dots, x_k, \dots, x_N) = e^{i\alpha\pi} \Psi(x_1, \dots, x_k, \dots, x_j, \dots, x_N)$$

and the behavior under particle exchange interpolates between bosons ($\alpha = 0$) and fermions ($\alpha = 1$). Of course the wave function is then in general not single-valued, and this description is not easy to use in practice. To describe free anyons, one way out is to realize that acting on Ψ with the free Hamiltonian

$$\sum_{j=1}^N (-\Delta_j + V(x_j))$$

is equivalent to acting on the bosonic wave function $\tilde{\Psi}$ with an effective, α -dependent Hamiltonian.

The Hamiltonian obtained in this way for N identical and ideal 2D anyons in a trapping potential $V : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ reads (see [15, 19, 20, 23, 39, 42, 53] for review and [34, 35, 36] for recent mathematical studies)

$$H_N := \sum_{j=1}^N ((p_j + \alpha \mathbf{A}_j)^2 + V(x_j)) \tag{1.1}$$

where

$$p_j = -i\nabla_j$$

is the usual momentum operator for particle j and (denoting $(x, y)^\perp := (-y, x)$)

$$\mathbf{A}_j := \sum_{k \neq j} \frac{(x_j - x_k)^\perp}{|x_j - x_k|^2} \quad (1.2)$$

is the (normalized¹) statistical gauge vector potential felt by particle j due to the influence of all other particles. The statistics parameter is denoted by α , corresponding to a statistical phase $e^{i\alpha\pi}$ under a continuous simple interchange of two particles. In this so-called “magnetic gauge picture”, 2D anyons are thus described as bosons, each of them carrying an Aharonov-Bohm magnetic flux of strength α .

We shall in this work assume

$$\alpha = \frac{\beta}{N-1} \rightarrow 0 \quad \text{when } N \rightarrow \infty, \quad (1.3)$$

where β is a given, fixed constant. We consider (1.1) as (formally) acting on the bosonic N -particle space $L^2_{\text{sym}}(\mathbb{R}^{2N})$, which together with the condition (1.3) means that we consider almost bosonic anyons. Note that if we simply took $\alpha \rightarrow 0$ at fixed N , we would recover ordinary bosons at leading order. One could then only see the effect of the non-trivial statistics in a perturbative expansion, a route followed e.g. in [48, 9, 49, 41, 10]. However, if $N \rightarrow \infty$ with fixed β as above, the anyon statistics has a leading order effect, manifest through a particular mean-field model with a self-consistent magnetic field of strength $\sim \beta$, studied e.g. in [6, 50, 51, 19, 52] and often called *the average field approximation* (see [53, 19] for review). Our aim in this work is to justify this description rigorously.

Actually, the Hamiltonian (1.1) is too singular to be considered as acting on a pure tensor product $u^{\otimes N} \in \bigotimes_{\text{sym}}^N \mathfrak{H}$, however regular the function u of the one-particle space $\mathfrak{H} \subseteq L^2(\mathbb{R}^2)$. We refer to [36, Section 2.1] for a discussion of the domain of (1.1), which requires the removal of the two-particle diagonals from the configuration space \mathbb{R}^{2N} . One way to circumvent this issue is to reintroduce a length scale R over which the magnetic charge is smeared. This so-called “extended anyons” model is discussed in [38, 50, 7], and is sometimes argued to be the correct physical description for anyons arising as quasi-particles in condensed-matter systems. In this paper we will allow R to become small when $N \rightarrow \infty$, in which case we recover the point-like anyons point of view, at least if one is willing to ignore the issue of non-commuting limits.

Let us consider the 2D Coulomb potential generated by a unit charge smeared over a disc of radius R :

$$w_R(x) := \log |\cdot| * \frac{\mathbb{1}_{B(0,R)}}{\pi R^2}(x). \quad (1.4)$$

Observing that (with the convention $w_0 := \log |\cdot|$)

$$\nabla^\perp w_0(x) = \frac{x^\perp}{|x|^2}, \quad \text{with } B_0(x) = \nabla^\perp \cdot \nabla^\perp w_0 = \Delta w_0 = 2\pi\delta_0,$$

the natural regularization of \mathbf{A}_j corresponding to the extended-anyons model is given by

$$\mathbf{A}_j^R := \sum_{k \neq j} \nabla^\perp w_R(x_j - x_k), \quad (1.5)$$

¹For increased clarity we will in this work separate α from \mathbf{A} , so that \mathbf{A} corresponds to the statistical vector potential of fermions modeled as bosons.

leading to the regularized Hamiltonian

$$H_N^R := \sum_{j=1}^N ((p_j + \alpha \mathbf{A}_j^R)^2 + V(x_j)). \quad (1.6)$$

We shall denote

$$E^R(N) := \inf \sigma(H_N^R) \quad (1.7)$$

the associated ground state energy (lowest eigenvalue) for N extended anyons.

For fixed $R > 0$ this operator is self-adjoint on $L_{\text{sym}}^2(\mathbb{R}^{2N})$ and one can even expand the squares to obtain a sum of terms that are all symmetric and relatively form-bounded with respect to the $\alpha = 0$ non-interacting operator². This gives

$$\begin{aligned} H_N^R &= \sum_{j=1}^N (p_j^2 + V(x_j)) \\ &\quad + \alpha \sum_{j \neq k} \left(p_j \cdot \nabla^\perp w_R(x_j - x_k) + \nabla^\perp w_R(x_j - x_k) \cdot p_j \right) \\ &\quad + \alpha^2 \sum_{j \neq k \neq \ell} \nabla^\perp w_R(x_j - x_k) \cdot \nabla^\perp w_R(x_j - x_\ell) \\ &\quad + \alpha^2 \sum_{j \neq k} |\nabla w_R(x_j - x_k)|^2. \end{aligned} \quad (1.8)$$

We also note that by the diamagnetic inequality (see, e.g., [29, Theorem 7.21] for $R > 0$, and [36, Lemma 4] for $R = 0$)

$$\langle \Psi, H_N^R \Psi \rangle \geq \langle |\Psi|, H_N(\alpha = 0) |\Psi| \rangle,$$

and hence $E^R(N) \geq N E_0$ for arbitrary α , with E_0 the ground state energy of the one-body operator $H_1 = p^2 + V$.

1.2. Average field approximation. The few-anyons problem can be studied within perturbation theory, yielding satisfactory information on the ground state and low-lying excitation spectrum [48, 9, 41, 10]. For many anyons however, it is hard to obtain results this way. A possible approximation to obtain a more tractable model when N is large consists in seeing the potential (1.2) or (1.5) as being independent of the precise positions x_j and instead generated by the mean distribution of the particles (whence the name, average field approximation [53])

$$\begin{aligned} \mathbf{A}[\rho] &:= \nabla^\perp w_0 * \rho, \\ \mathbf{A}^R[\rho] &:= \nabla^\perp w_R * \rho, \end{aligned} \quad (1.9)$$

where ρ is the one-body density (normalized in $L^1(\mathbb{R}^2)$) of a given bosonic wave function Ψ

$$\rho(x) = \int_{\mathbb{R}^{2(N-1)}} |\Psi(x, x_2, \dots, x_N)|^2 dx_2 \dots dx_N,$$

²By the boundedness of ∇w_R and using Cauchy-Schwarz, all terms are infinitesimally form-bounded in terms of $H_N(\alpha = 0)$ and hence H_N^R is a uniquely defined self-adjoint operator by the KLMN theorem [45, Theorem X.17]. We shall assume V is such that a form core is given by $C_c^\infty(\mathbb{R}^2)$.

say the ground-state wave function. One then obtains from (1.6) the approximate N -body Hamiltonian

$$H_N^{\text{af}}[\rho] := \sum_{j=1}^N \left((p_j + N\alpha \mathbf{A}^R[\rho])^2 + V(x_j) \right).$$

If one considers ρ as fixed, the ground state of this non-interacting magnetic Hamiltonian acting on $L_{\text{sym}}^2(\mathbb{R}^{2N})$ is a pure Bose condensate

$$\Psi_N = u^{\otimes N},$$

where $u \in L^2(\mathbb{R}^2)$ should minimize

$$\langle u | (p + N\alpha \mathbf{A}^R[\rho])^2 + V | u \rangle = N^{-1} \langle \Psi_N, H_N^{\text{af}}[\rho] \Psi_N \rangle.$$

For consistency, one should then impose that

$$|u|^2 = \rho,$$

which leads to a non-linear minimization problem. One thus looks for the minimum E^{af} and minimizer u^{af} of the following *average-field energy functional* (recall the notation $\beta \sim N\alpha$)

$$\mathcal{E}_R^{\text{af}}[u] := \int_{\mathbb{R}^2} \left(|(\nabla + i\beta \mathbf{A}^R[|u|^2]) u|^2 + V|u|^2 \right) \quad (1.10)$$

under the unit mass constraint

$$\int_{\mathbb{R}^2} |u|^2 = 1.$$

Note that, for this problem to be independent of N it is pretty natural to — in line with (1.3) — assume that $\beta \sim N\alpha$ is fixed. It is not difficult to see that if $N\alpha \rightarrow 0$ we recover at leading order a non-interacting theory, and we are back to the usual perturbation scheme. We should point out that the limiting functional $\mathcal{E}_{R=0}^{\text{af}}$, which defines a strictly two-dimensional model of particles with a self-generated magnetic field $B(x) = \text{curl } \beta \mathbf{A}[\rho](x) = 2\pi\beta\rho(x)$ without propagating degrees of freedom and to which one could further consider adding an external magnetic field, is also of relevance for various Chern-Simons formulations of anyonic theories (see, e.g., [55, 54, 19, 3]).

1.3. Average field versus mean field. In principle, the average field approximation does not require that the true ground state of H_N^R be Bose-condensed. In fact, the most common application of it has been in perturbing around fermions $\alpha = 1$ [13, 19, 50, 51, 52] (this has even been argued to be preferable [4, 53]), and usually one even restricts to the homogeneous setting with ρ a constant. However, the case of fixed $\beta \sim N\alpha$ which is natural for the study of (1.10), places the limit $N \rightarrow \infty$ of the original many-body problem in a mean-field-like regime for bosons. Indeed, observe that in (1.8), the two-body terms in the second line and the three-body term in the third line weigh a total $O(N)$ in the energy in this regime, comparable to the one-body term in the first line. The two-body term in the fourth line is of much smaller order, $O(1)$ roughly, which is fortunate because of its singularity. Actually, if one takes bluntly $R = 0$, the potential $|\nabla w_0|^2$ appearing in this term is not locally integrable, and hence an ansatz $\Psi_N = u^{\otimes N}$ would lead to an infinite energy. For extended anyons, $R > 0$ and this term can be safely dropped for leading order considerations.

The study of the regime (1.3) thus resembles a lot the usual mean-field limit for a large bosonic system (see [25, 26, 47] and references therein), but with important differences:

- The effective interaction is peculiar: it comprises a three-body term, and a two-body term which mixes position and momentum variables.
- The limit problem (1.10) comprises an effective self-consistent *magnetic* field. A term in the form of a self-consistent *electric* field is more usual.
- One should deal with the limit $R \rightarrow 0$ at the same time as $N \rightarrow \infty$, which is reminiscent of the NLS and GP limits for trapped Bose gases [32, 31, 30, 26, 40].

In order to make the analogy more transparent, we rewrite, for any normalized N -body bosonic wave function $\Psi_N \in L^2_{\text{sym}}(\mathbb{R}^{2N})$

$$\begin{aligned}
N^{-1} \langle \Psi_N | H_N^R | \Psi_N \rangle &= \text{Tr} \left[(p^2 + V) \gamma_N^{(1)} \right] \\
&+ \beta \text{Tr} \left[\left(p_1 \cdot \nabla^\perp w_R(x_1 - x_2) + \nabla^\perp w_R(x_1 - x_2) \cdot p_1 \right) \gamma_N^{(2)} \right] \\
&+ \beta^2 \frac{N-2}{N-1} \text{Tr} \left[\left(\nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) \right) \gamma_N^{(3)} \right] \\
&+ \beta^2 \frac{1}{N-1} \text{Tr} \left[|\nabla w_R(x_1 - x_2)|^2 \gamma_N^{(2)} \right], \tag{1.11}
\end{aligned}$$

where

$$\gamma_N^{(k)} := \text{Tr}_{k+1 \rightarrow N} [|\Psi_N\rangle\langle\Psi_N|]$$

is the k -body density matrix of the state $|\Psi_N\rangle\langle\Psi_N|$, normalized to have trace 1. The notation here means that we trace out the last $N - k$ variables from the integral kernel of $|\Psi_N\rangle\langle\Psi_N|$.

Since all terms at least at first sight weigh $O(1)$ or less, the folklore suggests to use an ansatz

$$\begin{aligned}
\Psi_N &= u^{\otimes N} \\
\gamma_N^{(k)} &= |u^{\otimes k}\rangle\langle u^{\otimes k}|. \tag{1.12}
\end{aligned}$$

Inserting this in the energy, dropping the last term, which is of order N^{-1} at least for fixed R , we obtain to leading order

$$N^{-1} \langle \Psi_N | H_N^R | \Psi_N \rangle \approx \mathcal{E}_R^{\text{af}}[u]. \tag{1.13}$$

Indeed, on the one hand,

$$\begin{aligned}
&\text{Tr} \left[\left(p_1 \cdot \nabla^\perp w_R(x_1 - x_2) + \nabla^\perp w_R(x_1 - x_2) \cdot p_1 \right) |u^{\otimes 2}\rangle\langle u^{\otimes 2}| \right] \\
&= i \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla \bar{u}(x) \bar{u}(y) \cdot \nabla^\perp w_R(x - y) u(x) u(y) dx dy \\
&- i \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \bar{u}(x) \bar{u}(y) \nabla^\perp w_R(x - y) \cdot \nabla u(x) u(y) dx dy \\
&= 2 \int_{\mathbb{R}^2} \mathbf{A}^R[|u|^2] \cdot \mathbf{J}[u], \tag{1.14}
\end{aligned}$$

using the definition (1.9) and denoting $\mathbf{J}[u]$ the current

$$\mathbf{J}[u] := \frac{i}{2} (u \nabla \bar{u} - \bar{u} \nabla u). \tag{1.15}$$

Note that this is really a phase current density:

$$\mathbf{J}[u] = \rho \nabla \varphi \quad \text{if} \quad u = \sqrt{\rho} e^{i\varphi}.$$

On the other hand

$$\begin{aligned} & \text{Tr} \left[\left(\nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) \right) \gamma_N^{(3)} \right] \\ &= \iiint_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} |u(x)|^2 |u(y)|^2 |u(z)|^2 \nabla^\perp w_R(x - y) \cdot \nabla^\perp w_R(x - z) \, dx dy dz \\ &= \int_{\mathbb{R}^2} |u|^2 |\mathbf{A}^R[|u|^2]|^2, \end{aligned} \tag{1.16}$$

and it suffices to combine these identities in (1.11) (and approximate $(N-2)/(N-1) \sim 1$) to obtain the desired expression (1.13) for the energy.

1.4. Main results. We may now state our main theorem, justifying the average field approximation in the almost-bosonic limit at the level of the ground state. For technical reasons we assume that the one-body potential is confining

$$V(x) \geq c|x|^s - C, \quad s > 0, \tag{1.17}$$

and that the size R of the extended anyons does not go to zero too fast in the limit $N \rightarrow \infty$. The rate we may handle depends on s . These assumptions are probably too restrictive from a physical point of view but our method of proof does not allow to relax them at present. Here and in the sequel, \mathcal{E}^{af} denotes the average-field functional (1.10) for $R = 0$, and E^{af} its infimum under a unit mass constraint. Although we do not state it explicitly, we could also keep R fixed when $N \rightarrow \infty$ and obtain the limit functional with finite R . The case of anyons in a bounded domain is also covered by our approach (modulo the discussion of boundary conditions) and the results in this case can be obtained by formally setting $s = \infty$ in the following.

Theorem 1.1 (Validity of the average field approximation).

We consider N extended anyons of radius $R \sim N^{-\eta}$ in an external potential V satisfying (1.17). We assume the relation

$$0 < \eta < \eta_0(s) := \frac{1}{4} \left(1 + \frac{1}{s} \right)^{-1}, \tag{1.18}$$

and that the statistics parameter scales as

$$\alpha = \beta / (N - 1)$$

for fixed $\beta \in \mathbb{R}$. Then, in the limit $N \rightarrow \infty$ we have for the ground-state energy

$$\frac{E^R(N)}{N} \rightarrow E^{\text{af}}. \tag{1.19}$$

Moreover, if Ψ_N is a sequence of ground states for H_N^R , with associated reduced density matrices $\gamma_N^{(k)}$, then modulo restricting to a subsequence we have

$$\gamma_N^{(k)} \rightarrow \int_{\mathcal{M}^{\text{af}}} |u^{\otimes k}\rangle \langle u^{\otimes k}| \, d\mu(u) \tag{1.20}$$

strongly in the trace-class when $N \rightarrow \infty$, where μ is a Borel probability measure supported on the set of minimizers of \mathcal{E}^{af} ,

$$\mathcal{M}^{\text{af}} := \{u \in L^2(\mathbb{R}^2) : \|u\|_{L^2} = 1, \mathcal{E}^{\text{af}}[u] = E^{\text{af}}\}.$$

Remark 1.2 (The size of the smearing radius).

As we said before, taking R not too small in the limit $N \rightarrow \infty$ is a requirement in our method of proof. It is likely that localization arguments could allow to take an s -independent $\eta < \eta_0(\infty) = 1/4$, corresponding to the rate we obtain for anyons in a bounded domain. To obtain an even better rate would require important new ideas.

It is in fact not clear whether some lower bound on R is a necessary condition for the average field description to be correct. For very small or zero R , it is still conceivable that a description in terms of a functional of the form of (1.10) is correct in the limit. Indeed, our above restriction stems from the method used to bound the ground-state energy from below, while for an upper bound the conditions on R (and even the finiteness of β) can be relaxed significantly. A further possibility would be to take short-range correlations into account via Jastrow factors, as in the GP limit for the usual Bose gas [32, 31, 30, 26, 40].

It would in any case be desirable to be able to take $R \ll N^{-1/2}$, the typical interparticle distance in our setting, because one could then argue that smearing the magnetic charges has very little effect. Even in the formal case $s = \infty$, Theorem 1.1 requires $R \gg N^{-1/4} \gg N^{-1/2}$, which is rather stringent. We nevertheless obtain the functional for point-like anyons in the limit.

It is sometimes argued in the literature [38, 50, 7] that for anyons arising as quasi-particles in condensed matter physics, the magnetic charges *should* be smeared over some length scale R . The relevant relation between R and N then depends on the context. \diamond

The rest of the paper contains the proof of Theorem 1.1. We start by collecting in Section 2 some operator bounds on the different terms of the N -body functional. This is required in order to have a correct control of the terms as a function of the kinetic energy in the limit $R \rightarrow 0$. For these estimates to be of use in the large- N limit we need an a priori bound on the kinetic energy of ground states of the N -body problem, also derived in Section 2. We deal with the mean-field limit in Section 3, using the method of [26]. Some important adaptations are required to deal with the anyonic Hamiltonian, and we focus on these. The goal here is to justify (with quantitative error bounds) the sensibility of the ansatz $\Psi_N = u^{\otimes N}$ when N becomes large, thus obtaining E_R^{af} as an approximation of the ground state energy per particle. The basic properties of the average-field functional (1.10) are worked out in Appendix A. In particular we study the limit $R \rightarrow 0$ to finally obtain E^{af} as an approximation of the many-body ground state energy per particle.

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2. THE EXTENDED ANYON HAMILTONIAN

In this section we give some bounds allowing to properly define and control the Hamiltonian (1.6). As previously mentioned, for extended anyons, it is possible to expand the Hamiltonian as in (1.8) and estimate it term by term. By the boundedness of the interaction it follows that H_N^R is defined uniquely as a self-adjoint operator on $L_{\text{sym}}^2(\mathbb{R}^{2N})$ with the same form domain as the non-interacting bosonic Hamiltonian

$$\sum_{j=1}^N (p_j^2 + V(x_j)).$$

However, in order to eventually take the limit $R \rightarrow 0$ we will need to deduce more precise bounds depending on R . These will be used to deal with the mean-field limit in Section 3.

In the following we introduce a fixed reference length scale $R_0 > 0$, and always assume $R \ll R_0$. Future constants, generically denoted by C , may implicitly depend on R_0 .

2.1. Operator bounds for the interaction terms. We start with some estimates on the different terms in (1.11), exploiting the regularizing effect of taking $R > 0$. The following is standard:

Lemma 2.1 (The smeared Coulomb potential).

Let w_R be defined as in (1.4). There is a constant $C > 0$ depending only on R_0 such that

$$\sup_{B(0, R_0)} |w_R| \leq C + |\log R|, \quad \sup_{\mathbb{R}^2} |\nabla w_R| \leq \frac{C}{R}, \quad \sup_{B(0, R_0)^c} |\nabla w_R| \leq C. \quad (2.1)$$

Moreover, for any $2 < p < \infty$,

$$\|\nabla w_R\|_{L^p(\mathbb{R}^2)} \leq C_p R^{2/p-1}. \quad (2.2)$$

Proof. A simple application of Newton's theorem [29, Theorem 9.7] yields

$$w_R(x) = \begin{cases} \log |x| & \text{if } |x| \geq R \\ \log R + \frac{1}{2} \left(\frac{|x|^2}{R^2} - 1 \right) & \text{if } 0 \leq |x| \leq R, \end{cases} \quad \nabla w_R(x) = \begin{cases} x/|x|^2 & \text{if } |x| \geq R \\ x/R^2 & \text{if } 0 \leq |x| \leq R, \end{cases} \quad (2.3)$$

and (2.1) clearly follows. For (2.2) we compute

$$\|\nabla w_R\|_{L^p(\mathbb{R}^2)}^p = 2\pi \int_0^R \frac{r^p}{R^{2p}} r dr + 2\pi \int_R^\infty r^{-p} r dr \leq C_p^p R^{2-p},$$

where $C_p > 0$ depends only on $p > 2$. □

We first estimate the most singular term of the Hamiltonian, corresponding to the fourth line of (1.8). Since it comes with a relative weight $O(N^{-1})$ in the total energy, the following bound will be enough to discard it from leading order considerations.

Lemma 2.2 (Singular two-body term).

We have that, as operators on $L^2(\mathbb{R}^4)$ or $L_{\text{sym}}^2(\mathbb{R}^4)$,

$$|\nabla w_R(x - y)|^2 \leq C_\varepsilon R^{-\varepsilon} (p_x^2 + 1) \quad (2.4)$$

for any $\varepsilon > 0$.

Proof. We start with a well-known simple application of Hölder's and Sobolev's inequalities: for any $W : \mathbb{R}^2 \mapsto \mathbb{R}$ and $f \in C_c^\infty(\mathbb{R}^4)$

$$\begin{aligned} \langle f | W(x-y) | f \rangle &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \overline{f(x,y)} W(x-y) f(x,y) dx dy \\ &\leq \|W\|_{L^p} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |f(x,y)|^{2q} dx \right)^{1/q} dy \\ &\leq C \|W\|_{L^p} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (|\nabla_x f(x,y)|^2 + |f(x,y)|^2) dx dy \\ &= C \|W\|_{L^p} \langle f | (-\Delta_x + 1) \otimes \mathbb{1} | f \rangle \end{aligned} \quad (2.5)$$

where we may take any $p > 1$, $q = \frac{p}{p-1} \in (1, +\infty)$, and we use that in \mathbb{R}^2 , for any $1 \leq q < \infty$

$$\|g\|_{L^{2q}}^2 \leq C_q \left(\|\nabla g\|_{L^2}^2 + \|g\|_{L^2}^2 \right),$$

see, e.g., [29, Theorem 8.5 ii]. Next we may use (2.2) with $W = |\nabla w_R|^2$ and $p = 1 + \varepsilon'$ to conclude

$$\|W\|_{L^p} = \|\nabla w_R\|_{L^{2p}}^2 \leq C_{2p}^2 R^{2/p-2} \leq C_\varepsilon R^{-\varepsilon},$$

with a constant $C_\varepsilon > 0$ for given $\varepsilon > 0$. \square

We next deal with the two-body term mixing position and momentum, second line of (1.8). This is somehow the most difficult term to handle, and it is crucial to observe that it acts on the current and not on the full momentum. We shall use three different bounds. In the following lemma, (2.6) has a worst R -dependence but it behaves better for large momenta than (2.7) and (2.8), a fact that will be useful when projecting the problem onto finite dimensional spaces in the next section. Estimate (2.8) might seem a bit better than (2.7), but we will actually need a bound on the absolute value in the sequel, which is not provided by (2.8).

Lemma 2.3 (Mixed two-body term).

For $R < R_0$ small enough we have that, as operators on $L_{\text{sym}}^2(\mathbb{R}^4)$,

$$\left| p_x \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x \right| \leq C R^{-1} |p_x|, \quad (2.6)$$

$$\left| p_x \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x \right| \leq C_\varepsilon R^{-\varepsilon} (p_x^2 + 1), \text{ for all } \varepsilon > 0, \quad (2.7)$$

and

$$\pm \left(p_x \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x \right) \leq C(1 + |\log R|) (p_x^2 + 1). \quad (2.8)$$

Proof. The bounds (2.6) and (2.7) are based on the same basic computation.

Proof of (2.6). First note that

$$p_x \cdot \nabla^\perp w_R(x-y) = \nabla^\perp w_R(x-y) \cdot p_x \quad (2.9)$$

because $\nabla_x \cdot \nabla^\perp w_R(x-y) = 0$. We can then square the expression we want to estimate, obtaining

$$\left(p_x \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x \right)^2 = 4 p_x \cdot \nabla^\perp w_R(x-y) \nabla^\perp w_R(x-y) \cdot p_x.$$

Consequently, for any $f = f(x, y) \in C_c^\infty(\mathbb{R}^4)$,

$$\begin{aligned} & \left| \left\langle f \left| \left(p_x \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x \right)^2 \right| f \right\rangle \right| \\ &= 4 \left| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\nabla_x \bar{f}(x, y) \cdot \nabla^\perp w_R(x-y) \right) \left(\nabla_x f(x, y) \cdot \nabla^\perp w_R(x-y) \right) dx dy \right| \\ &\leq 4 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |\nabla_x f(x, y)|^2 \left| \nabla^\perp w_R(x-y) \right|^2 dx dy. \end{aligned}$$

Inserting (2.1) we get

$$\left| \left\langle f \left| \left(p_x \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x \right)^2 \right| f \right\rangle \right| \leq \frac{C}{R^2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |\nabla_x f|^2 dx dy$$

and thus

$$\left(p_x \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x \right)^2 \leq \frac{C}{R^2} p_x^2.$$

We deduce (2.6) because the square root is operator monotone (see, e.g., [2, Chapter 5]).

Proof of (2.7). We proceed in the same way but use Lemma 2.2 instead of just the rough bound (2.1) (we denote $x = (x_1, x_2) \in \mathbb{R}^2$):

$$\begin{aligned} & \left| \left\langle f \left| \left(p_x \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x \right)^2 \right| f \right\rangle \right| \\ &\leq 4 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |\nabla_x f(x, y)|^2 \left| \nabla^\perp w_R(x-y) \right|^2 dx dy \\ &= \left\langle \partial_{x_1} f, \left| \nabla^\perp w_R(x-y) \right|^2 \partial_{x_1} f \right\rangle_{L^2(\mathbb{R}^4)} + \left\langle \partial_{x_2} f, \left| \nabla^\perp w_R(x-y) \right|^2 \partial_{x_2} f \right\rangle_{L^2(\mathbb{R}^4)} \\ &\leq \frac{C_\varepsilon}{R^\varepsilon} (\langle \partial_{x_1} f, (-\Delta_x + 1) \partial_{x_1} f \rangle + \langle \partial_{x_2} f, (-\Delta_x + 1) \partial_{x_2} f \rangle) \leq \frac{C_\varepsilon}{R^\varepsilon} \langle f, (-\Delta_x + 1)^2 f \rangle. \end{aligned}$$

Thus

$$\left(p_x \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x \right)^2 \leq \frac{C_\varepsilon}{R^\varepsilon} (p_x^2 + 1)^2$$

for any $\varepsilon > 0$, and the desired bound again follows by taking the square root.

Proof of (2.8). The idea is here a bit different. We pick $f \in C_c^\infty(\mathbb{R}^4; \mathbb{C})$ and compute as in (1.14)

$$\left\langle f \left| p_x \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x \right| f \right\rangle = 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla^\perp w_R(x-y) \cdot \mathbf{J}_x[f] dx dy$$

with

$$\mathbf{J}_x[f] = \frac{i}{2} (f \overline{\nabla_x f} - \overline{f} \nabla_x f).$$

We then split this according to a partition of unity $\chi + \eta = 1$ where $\chi \equiv 1$ in the ball $B(0, R_0)$ and $\eta \equiv 1$ outside of the ball $B(0, 2R_0)$:

$$\begin{aligned} \left\langle f | p_x \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x | f \right\rangle &= 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla^\perp (\chi(x-y)w_R(x-y)) \cdot \mathbf{J}_x[f] dx dy \\ &\quad + 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla^\perp (\eta(x-y)w_R(x-y)) \cdot \mathbf{J}_x[f] dx dy. \end{aligned}$$

To control the χ term we use Stokes' formula and deduce

$$2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla^\perp (\chi(x-y)w_R(x-y)) \cdot \mathbf{J}_x[f] dx dy = -2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \chi(x-y)w_R(x-y) \operatorname{curl}_x \mathbf{J}_x[f] dx dy.$$

It is easy to see that

$$|\operatorname{curl}_x \mathbf{J}_x[f]| \leq |\nabla_x f|^2$$

pointwise, see e.g. [11, Lemma 3.4]. We thus obtain

$$\begin{aligned} &\pm 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla^\perp (\chi(x-y)w_R(x-y)) \cdot \mathbf{J}_x[f] dx dy \\ &\leq 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |\chi(x-y)| |w_R(x-y)| |\nabla_x f(x,y)|^2 dx dy \\ &\leq C(1 + |\log R|) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |\nabla_x f(x,y)|^2 dx dy \end{aligned}$$

in view of (2.1). For the η term we note that

$$|\mathbf{J}_x[f]| \leq |f| |\nabla_x f|.$$

Thus

$$\begin{aligned} &\pm 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla^\perp (\eta(x-y)w_R(x-y)) \cdot \mathbf{J}_x[f] dx dy \\ &\leq \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left| \nabla^\perp (\eta(x-y)w_R(x-y)) \right|^2 |f(x,y)|^2 dx dy + \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |\nabla_x f(x,y)|^2 dx dy \\ &\leq C \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \bar{f}(-\Delta_x + 1) f dx dy. \end{aligned}$$

For the first term we used that (2.3) implies

$$\left| \nabla^\perp (\eta w_R) \right| \leq |\nabla \eta| |w_R| + |\eta| |\nabla w_R| \leq C$$

because $\eta \equiv 0$ in $B(0, R_0)$ and $\eta \equiv 1$ outside of $B(0, 2R_0)$. Gathering these estimates we obtain the desired operator bound. \square

The three-body term (third line of (1.8)) is actually a pretty regular potential term, as shown in the following:

Lemma 2.4 (Three-body term).

We have that, as operators on $L^2_{\text{sym}}(\mathbb{R}^6)$,

$$0 \leq \nabla^\perp w_R(x-y) \cdot \nabla^\perp w_R(x-z) \leq C(p_x^2 + 1). \quad (2.10)$$

The essential ingredient of the proof is the following three-particle Hardy inequality of [18, Lemma 3.6] (see also [33] for relevant methods and generalizations):

Lemma 2.5 (Three-body Hardy inequality).

Let $d \geq 2$ and $u : \mathbb{R}^{3d} \rightarrow \mathbb{C}$. Let $\mathcal{R}(x, y, z)$ be the circumradius of the triangle with vertices $x, y, z \in \mathbb{R}^d$, and $\rho(x, y, z) := \sqrt{|x - y|^2 + |y - z|^2 + |z - x|^2}$. Then $\mathcal{R}^{-2} \leq 9\rho^{-2}$ pointwise, and

$$3(d-1)^2 \int_{\mathbb{R}^{3d}} \frac{|u(x, y, z)|^2}{\rho(x, y, z)^2} dx dy dz \leq \int_{\mathbb{R}^{3d}} \left(|\nabla_x u|^2 + |\nabla_y u|^2 + |\nabla_z u|^2 \right) dx dy dz. \quad (2.11)$$

Proof of Lemma 2.4. Since we consider the operator as acting on symmetric wave functions it is equivalent to estimate

$$\sum_{\text{cyclic in } x, y, z} \nabla^\perp w_R(x - y) \cdot \nabla^\perp w_R(x - z). \quad (2.12)$$

In general, let $x, y, z \in \mathbb{R}^d$ denote the vertices of a triangle and $|x|_R := \max\{|x|, R\}$ a regularized distance. Then we claim the following geometric fact:

$$0 \leq \sum_{\text{cyclic in } x, y, z} \frac{x - y}{|x - y|_R^2} \cdot \frac{x - z}{|x - z|_R^2} \leq \frac{C}{\rho(x, y, z)^2}. \quad (2.13)$$

Recalling (2.3) this gives a control on the expression we are interested in. Equivalently, we shall prove that

$$\begin{aligned} 0 &\leq |y - z|_R^2 (x - y) \cdot (x - z) + |z - x|_R^2 (y - z) \cdot (y - x) + |x - y|_R^2 (z - x) \cdot (z - y) \\ &\leq C \frac{|x - y|_R^2 |y - z|_R^2 |z - x|_R^2}{|x - y|^2 + |y - z|^2 + |z - x|^2}, \end{aligned} \quad (2.14)$$

for some constant $C > 0$ independent of R .

Let us consider each of the different geometric configurations that may occur. In the case that all edge lengths of the triangle are greater than R , the cyclic expression that we wish to estimate in (2.13) reduces to $\frac{1}{2}\mathcal{R}(x, y, z)^{-2}$ (see [18, Lemma 3.2]), which is clearly non-negative and bounded by $\frac{9}{2}\rho^{-2}$ from Lemma 2.5. On the other hand, if all edge lengths are smaller than R then the expression equals $\frac{1}{2R^4}\rho^2$ (cf. [18, Lemma 3.4]), for which we have $0 \leq \frac{1}{2R^4}\rho^2 \leq \frac{9}{2}\rho^{-2}$ since $\rho^2 \leq 3R^2$. If two of the edges are short and one long, say $|x - y|, |y - z| \leq R$ and $|z - x| \geq R$, then the expression to be estimated in (2.14) reads

$$\begin{aligned} &R^2(x - y) \cdot (x - z) + |z - x|^2(y - z) \cdot (y - x) + R^2 \underbrace{(z - x) \cdot (z - y)}_{-(z - y) \cdot (x - z)} \\ &= R^2((x - y) - (z - y)) \cdot (x - z) + |x - z|^2(y - z) \cdot (y - x) \\ &= |x - z|^2 \left(R^2 + (y - z) \cdot (y - x) \right) \geq |x - z|^2(R^2 - R^2) \geq 0. \end{aligned}$$

We furthermore have the upper bound

$$|x - z|^2 \left(R^2 + (y - z) \cdot (y - x) \right) \leq 2R^2|x - z|^2,$$

while the r.h.s. of (2.14) is larger than

$$\frac{R^4|x - z|^2}{2R^2 + |x - z|^2} \geq \frac{1}{6}R^2|x - z|^2,$$

using that $|x - z| \leq |x - y| + |y - z| \leq 2R$.

This leaves the case that only one edge is short, say $|x - y| \leq R$, and the others long, $|y - z|, |z - x| \geq R$. We thus consider the expression in (2.14)

$$|y - z|^2 (x - y) \cdot (x - z) + |z - x|^2 (y - z) \cdot (y - x) + R^2 (z - x) \cdot (z - y). \quad (2.15)$$

We will here use methods from [33], namely the geometric (Clifford) algebra $\mathcal{G}(\mathbb{R}^d)$ over \mathbb{R}^d (see [37] for a general introduction). In the case $d = 2$ or $d = 3$ one can think of this as the real algebra generated by the Pauli matrices σ_j , with scalar projection $\langle A \rangle_0 := \frac{1}{2} \text{Tr } A$ and the embedding of scalars (0-vectors) $1 \hookrightarrow \mathbf{1}$ and of 1-vectors $\mathbb{R}^d \ni x \hookrightarrow \sum_{j=1}^d x_j \sigma_j \in \mathcal{G}(\mathbb{R}^d)$, and with the product of two 1-vectors $xy = x \cdot y + x \wedge y$ decomposing into a traceful symmetric scalar part and a traceless antisymmetric bivector part. We have then, using tracelessness of the bivector parts of such products and the linearity and cyclicity of the trace,

$$\begin{aligned} & |y - z|^2 (x - y) \cdot (x - z) + |z - x|^2 (y - z) \cdot (y - x) \\ &= \langle (y - z)^2 (x - y)(x - z) + (z - x)^2 (y - z)(y - x) \rangle_0 \\ &= \langle (y - z)(x - y)(x - z)(y - z) + (y - x)(z - x)(z - x)(y - z) \rangle_0 \\ &= \langle (y - z)(x - y)(z - x)(z - y) + (x - y)(z - x)(z - x)(z - y) \rangle_0 \\ &= \left\langle \left((y - z)(x - y) + (z - x)(x - y) + 2(x - y) \wedge (z - x) \right) (z - x)(z - y) \right\rangle_0 \\ &= \langle (y - x)(x - y)(z - x)(z - y) \rangle_0 + 2 \langle (x - y) \wedge (z - x)(z - x)(z - y) \rangle_0 \\ &= - \langle (x - y)^2 (z - x)(z - y) \rangle_0 + 2 \underbrace{\langle (x - y) \wedge (z - x)(z - x) \wedge (z - y) \rangle_0}_{z-y-(z-x)} \\ &= -|x - y|^2 (z - x) \cdot (z - y) + 2 \langle B^\dagger B \rangle_0, \end{aligned}$$

with $B := (z - x) \wedge (z - y)$ and its Hermite conjugate $B^\dagger = (z - y) \wedge (z - x)$. In the fourth and fifth steps we used $xy = yx + 2x \wedge y$ for the second term and then $(y - z) + (z - x) = y - x$, while for the final steps we again used the properties of the trace and that $B^\dagger B = |B|^2$ is scalar. Thus, the expression (2.15) we wish to estimate equals

$$(R^2 - |x - y|^2) (z - x) \cdot (z - y) + 2|B|^2 \geq 0,$$

where for the lower bound we also used that $(z - x) \cdot (z - y) \geq 0$ since $x - y$ is the shortest edge. For an upper bound we can use permutation invariance (cf. [33, Proposition 15]) of

$$|B| = |(x - y) \wedge (x - z)| \leq |x - y||x - z|,$$

and for example that $|y - z| \leq R + |x - z| \leq 2|x - z|$. Hence

$$(R^2 - |x - y|^2) (z - x) \cdot (z - y) + 2|B|^2 \leq 4R^2|x - z|^2,$$

while for the r.h.s. of (2.14), with analogously $|x - z| \leq 2|y - z|$,

$$\frac{R^2|y - z|^2|z - x|^2}{R^2 + |y - z|^2 + |z - x|^2} \geq \frac{R^2|y - z|^2|z - x|^2}{6|y - z|^2} = \frac{1}{6}R^2|x - z|^2.$$

We also remark that the non-negativity of (2.13) is in general false if $|\cdot|_R$ is replaced by an arbitrary radial function, as can be checked when taking e.g. $|x|_R = e^{|x|^2/2}$.

Finally, the estimate (2.10) follows simply by applying Lemma 2.5 with $d = 2$ to (2.13) and using the symmetry of functions in $L_{\text{sym}}^2(\mathbb{R}^6)$. \square

2.2. A priori bound for the ground state. For the estimates of the previous subsection to apply efficiently, we need an a priori bound on ground states (or approximate ground states) of the N -body Hamiltonian (1.6), provided in the following:

Proposition 2.6 (A priori bound for many-body ground states).

Let $\Psi_N \in L^2_{\text{sym}}(\mathbb{R}^{2N})$ be a (sequence of) approximate ground states for H_N^R , that is,

$$\langle \Psi_N, H_N^R \Psi_N \rangle \leq E^R(N)(1 + o(1)) \text{ when } N \rightarrow \infty.$$

Denote by $\gamma_N^{(1)}$ the associated sequence of one-body density matrices. In the regime (1.3), assuming a bound $R \geq N^{-\eta}$ for some $\eta > 0$ independent of N , we have

$$\text{Tr} \left[(p^2 + V) \gamma_N^{(1)} \right] \leq C(1 + \beta^2), \quad (2.16)$$

where C is a constant independent of β , N and R .

Proof. We proceed in two steps.

Step 1. Using a trial state $u^{\otimes N}$ with $u = |u| \in C_c^\infty(\mathbb{R}^2)$, we easily obtain from (1.11) and the above bounds (note that the R -divergent mixed two-body term is zero on such a u , and that the singular two-body term gives a lower-order contribution)

$$E^R(N) \leq C(1 + \beta^2)N. \quad (2.17)$$

Next we use the diamagnetic inequality [29, Theorem 7.21] in each variable to obtain

$$\begin{aligned} \langle \Psi_N, H_N^R \Psi_N \rangle &= \sum_{j=1}^N \int_{\mathbb{R}^{2N}} \left(|(-i\nabla_j + \alpha \mathbf{A}_j^R) \Psi_N|^2 + V(x_j) |\Psi_N|^2 \right) dx_1 \dots dx_N \\ &\geq \sum_{j=1}^N \int_{\mathbb{R}^{2N}} \left(|\nabla_j \Psi_N|^2 + V(x_j) |\Psi_N|^2 \right) dx_1 \dots dx_N. \end{aligned}$$

We deduce the bound

$$\text{Tr} \left[(p^2 + V) \gamma_{N,+}^{(1)} \right] \leq C(1 + \beta^2), \quad (2.18)$$

where we denote

$$\gamma_{N,+}^{(k)} := \text{Tr}_{k+1 \rightarrow N} [|\Psi_N\rangle \langle \Psi_N|]$$

the reduced k -body density matrix of $|\Psi_N\rangle$.

Step 2. Next we expand the Hamiltonian and use the Cauchy-Schwarz inequality for operators to obtain

$$\begin{aligned} H_N^R &= \sum_{j=1}^N (p_j^2 + \alpha p_j \cdot \mathbf{A}_j^R + \alpha \mathbf{A}_j^R \cdot p_j + \alpha^2 |\mathbf{A}_j^R|^2 + V(x_j)) \\ &\geq \sum_{j=1}^N ((1 - 2\delta^{-1})p_j^2 + (1 - 2\delta)\alpha^2 |\mathbf{A}_j^R|^2 + V(x_j)) \\ &= \sum_{j=1}^N \left(\frac{1}{2}(p_j^2 + V(x_j)) - 7 \frac{\beta^2}{(N-1)^2} |\mathbf{A}_j^R|^2 \right), \end{aligned}$$

choosing $\delta = 4$. Thus, using (2.17) we have

$$\mathrm{Tr} \left[(p^2 + V) \gamma_N^{(1)} \right] \leq C(1 + \beta^2) + \frac{C\beta^2}{N(N-1)^2} \left\langle \Psi_N, \sum_{j=1}^N |\mathbf{A}_j^R|^2 \Psi_N \right\rangle. \quad (2.19)$$

Then, since the last term in the right-hand side is purely a potential term

$$\left\langle \Psi_N, \sum_{j=1}^N |\mathbf{A}_j^R|^2 \Psi_N \right\rangle = \left\langle |\Psi_N|, \sum_{j=1}^N |\mathbf{A}_j^R|^2 |\Psi_N| \right\rangle.$$

We then expand the squares as in (1.11), and use Lemmas 2.2 and 2.4 to obtain for any $\varepsilon > 0$

$$\begin{aligned} \frac{1}{N(N-1)^2} \left\langle |\Psi_N|, \sum_{j=1}^N |\mathbf{A}_j^R|^2 |\Psi_N| \right\rangle &\leq C \mathrm{Tr} \left[\nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) \gamma_{N,+}^{(3)} \right] \\ &\quad + C N^{-1} \mathrm{Tr} \left[|\nabla w_R(x_1 - x_2)|^2 \gamma_{N,+}^{(2)} \right] \\ &\leq C \mathrm{Tr} \left[(p_1^2 + 1) \otimes \mathbb{1} \otimes \mathbb{1} \gamma_{N,+}^{(3)} \right] + C_\varepsilon R^{-\varepsilon} N^{-1} \mathrm{Tr} \left[(p_1^2 + 1) \otimes \mathbb{1} \gamma_{N,+}^{(2)} \right] \\ &\leq C (1 + C_\varepsilon N^{-1} R^{-\varepsilon}) \mathrm{Tr} \left[(p_1^2 + 1) \gamma_{N,+}^{(1)} \right]. \end{aligned}$$

Inserting the estimate (2.18) and recalling that we assume $R \geq N^{-\eta}$ we conclude the proof by going back to (2.19). \square

3. MEAN-FIELD LIMIT

We now turn to the study of the mean-field limit per se. The strategy is the same as in [26], but the peculiarities of the anyon Hamiltonian add some important twists, and we shall rely heavily on the estimates of the preceding section.

3.1. Preliminaries. We first recall some constructions from [24, 26].

Energy cut-off. We denote by P the spectral projector of $-\Delta + V$ below a given (large) energy cut-off Λ that we shall optimize over in the end:

$$P := \mathbb{1}_{h \leq \Lambda}, \quad h = -\Delta + V. \quad (3.1)$$

Let

$$N_\Lambda = \dim(PL^2(\mathbb{R}^2))$$

be the number of energy levels obtained this way, and recall the following Cwikel-Lieb-Rozenblum type inequality, proved by well-known methods, as in [26, Lemma 3.3]:

Lemma 3.1 (Number of energy levels below the cut-off).

For Λ large enough we have

$$N_\Lambda \leq C \Lambda^{1+2/s}. \quad (3.2)$$

We shall also denote

$$Q = \mathbb{1} - P$$

the orthogonal projector onto excited energy levels.

Localization in Fock space. We quickly recall the procedure of geometric localization, following the notation of [24]. Let γ_N be an arbitrary N -body (mixed) state. Associated with the given projector P , there is a localized state G_N^P in the Fock space

$$\mathcal{F}(\mathfrak{H}) = \mathbb{C} \oplus \mathfrak{H} \oplus \mathfrak{H}^2 \oplus \cdots$$

of the form

$$G_N^P = G_{N,0}^P \oplus G_{N,1}^P \oplus \cdots \oplus G_{N,N}^P \oplus 0 \oplus \cdots \quad (3.3)$$

with the property that its reduced density matrices satisfy

$$P^{\otimes n} \gamma_N^{(n)} P^{\otimes n} = (G_N^P)^{(n)} = \binom{N}{n}^{-1} \sum_{k=n}^N \binom{k}{n} \text{Tr}_{n+1 \rightarrow k} [G_{N,k}^P] \quad (3.4)$$

for any $0 \leq n \leq N$. Here we use the convention that

$$\gamma_N^{(n)} := \text{Tr}_{n+1 \rightarrow N} [\gamma_N],$$

which differs from the convention of [24], whence the different numerical factors in (3.4). We also have a localized state G_N^Q corresponding to the projector Q , which is defined similarly.

The relations (3.4) determine the localized states G_N^P, G_N^Q uniquely and they ensure that G_N^P and G_N^Q are (mixed) states on the projected Fock spaces $\mathcal{F}(P\mathfrak{H})$ and $\mathcal{F}(Q\mathfrak{H})$, respectively:

$$\sum_{k=0}^N \text{Tr} [G_{N,k}^{P/Q}] = 1. \quad (3.5)$$

de Finetti measure for the projected state. We will apply the quantitative de Finetti Theorem in finite dimensional spaces of [8, 5, 17, 27] to the localized state G_N^P , in order to approximate its three-body density matrix. The following is the equivalent of [26, Lemma 3.4] and the proof is exactly similar:

Lemma 3.2 (Quantitative quantum de Finetti for the localized state).

Let γ_N be an arbitrary N -body (mixed) state. Define

$$d\mu_N(u) := \sum_{k=3}^N \binom{N}{3}^{-1} \binom{k}{3} d\mu_{N,k}(u), \quad d\mu_{N,k}(u) := \dim(P\mathfrak{H})_{\text{sym}}^k \langle u^{\otimes k}, G_{N,k}^P u^{\otimes k} \rangle du \quad (3.6)$$

and

$$\tilde{\gamma}_N^{(3)} := \int_{SP\mathfrak{H}} |u^{\otimes 3}\rangle \langle u^{\otimes 3}| d\mu_N(u). \quad (3.7)$$

Then there is a constant $C > 0$ such that for every $N \in \mathbb{N}$ and $\Lambda > 0$, we have

$$\text{Tr} \left| P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3} - \tilde{\gamma}_N^{(3)} \right| \leq \frac{CN_\Lambda}{N}. \quad (3.8)$$

3.2. Truncated Hamiltonian. For an energy lower bound we are first going to roughly bound some terms in the Hamiltonian. Let us introduce the effective three-body Hamiltonian

$$\begin{aligned} \tilde{H}_3^R := \frac{1}{3}(h_1 + h_2 + h_3) + \frac{\beta}{6} \sum_{1 \leq j \neq k \leq 3} & \left(p_j \cdot \nabla^\perp w_R(x_j - x_k) + \nabla^\perp w_R(x_j - x_k) \cdot p_j \right) \\ & + \beta^2 \nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) \end{aligned} \quad (3.9)$$

where h_i is understood to act on the i -th variable (recall that $h = -\Delta + V$). For shortness we denote

$$W_2 = p_1 \cdot \nabla^\perp w_R(x_1 - x_2) + \nabla^\perp w_R(x_1 - x_2) \cdot p_1$$

the two-body part of \tilde{H}_3^R , and

$$W_3 = \nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3)$$

its three-body part. With this notation

$$\tilde{H}_3^R := \frac{1}{3}(h_1 + h_2 + h_3) + \frac{\beta}{6} \sum_{1 \leq i \neq j \leq 3} W_2(i, j) + \beta^2 W_3$$

where $W_2(i, j)$ acts on variables i and j . Also note that for $\|u\| = 1$, by (1.14), (1.16),

$$\langle u^{\otimes 3}, \tilde{H}_3^R u^{\otimes 3} \rangle = \mathcal{E}_R^{\text{af}}[u] \geq E_R^{\text{af}}.$$

We bound the full energy from below in terms of a projected version of \tilde{H}_3^R :

Proposition 3.3 (Truncated three-body Hamiltonian).

Let Ψ_N be a (sequence of) approximate ground state(s) for H_N^R with associated reduced density matrices $\gamma_N^{(k)}$. Then, for any $\varepsilon > 0$ and R small enough,

$$\begin{aligned} \frac{1}{N} \langle \Psi_N, H_N^R \Psi_N \rangle & \geq \text{Tr} \left[\tilde{H}_3^R P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3} \right] + C_\beta \Lambda \text{Tr} [Q \gamma_N^{(1)}] \\ & - C_\beta \left(\frac{1}{N} + \frac{C_\varepsilon}{\sqrt{\Lambda} R^{1+\varepsilon}} + \frac{1}{\Lambda R^2} \right). \end{aligned} \quad (3.10)$$

Proof. We proceed in several steps.

Step 1. We first claim that

$$\frac{1}{N} \langle \Psi_N, H_N^R \Psi_N \rangle \geq \text{Tr} \left[\tilde{H}_3^R \gamma_N^{(3)} \right] - C_\beta N^{-1}. \quad (3.11)$$

To see this, we start from (1.11). For a lower bound we drop the term on the fourth line, which is positive. Then one only has to correct the N -dependent factors in front of the third line. The term we have to drop to obtain (3.11) is bounded as

$$\beta^2 \left| 1 - \frac{N-2}{N-1} \right| \left| \text{Tr} \left[\left(\nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) \right) \gamma_N^{(3)} \right] \right| \leq C_\beta N^{-1}$$

upon using the a priori bound (2.16) combined with (2.10).

Step 2. We next proceed to bound the right-hand side of (3.11) from below in terms of a localized version of \tilde{H}_3^R and remainder terms to be estimated in the next step. We shall

need the projectors

$$\begin{aligned}\Pi_2 &= \mathbb{1}^{\otimes 2} - P^{\otimes 2} \\ \Pi_3 &= \mathbb{1}^{\otimes 3} - P^{\otimes 3}\end{aligned}$$

and make a repeated use of the inequality

$$ABC + CBA \geq -\varepsilon A|B|A - \varepsilon^{-1}C|B|C, \quad \varepsilon > 0, \quad (3.12)$$

for any self-adjoint operators A, B, C .

We claim that

$$\begin{aligned}\mathrm{Tr} \left[\tilde{H}_3^R \gamma_N^{(3)} \right] &\geq \mathrm{Tr} \left[\tilde{H}_3^R P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3} \right] + \mathrm{Tr} \left[h Q \gamma_N^{(1)} Q \right] \\ &\quad - |\beta|(3 + \delta_1) \mathrm{Tr} \left[\Pi_2 |W_2| \Pi_2 \gamma_N^{(2)} \right] - |\beta| \delta_1^{-1} \mathrm{Tr} \left[P^{\otimes 2} |W_2| P^{\otimes 2} \gamma_N^{(2)} \right] \\ &\quad - 2|\beta| \mathrm{Tr} \left[P^{\otimes 2} \otimes Q |W_2(1, 2)| P^{\otimes 2} \otimes Q \gamma_N^{(3)} \right] \\ &\quad - \beta^2(1 + \delta_2) \mathrm{Tr} \left[\Pi_3 |W_3| \Pi_3 \gamma_N^{(3)} \right] - \beta^2 \delta_2^{-1} \mathrm{Tr} \left[P^{\otimes 3} |W_3| P^{\otimes 3} \gamma_N^{(3)} \right] \quad (3.13)\end{aligned}$$

where δ_1 and δ_2 are two positive parameters to be chosen later on.

To prove (3.13), first note that

$$\mathrm{Tr} \left[\tilde{H}_3^R \gamma_N^{(3)} \right] = \mathrm{Tr} \left[h \gamma_N^{(1)} \right] + \frac{\beta}{2} \mathrm{Tr} \left[W_2 \gamma_N^{(2)} \right] + \beta^2 \mathrm{Tr} \left[W_3 \gamma_N^{(3)} \right].$$

Then, for the one-body term we have

$$\begin{aligned}\mathrm{Tr} \left[h \gamma_N^{(1)} \right] &= \mathrm{Tr} \left[P h P \gamma_N^{(1)} \right] + \mathrm{Tr} \left[Q h Q \gamma_N^{(1)} \right] \\ &\geq \frac{1}{3} \mathrm{Tr} \left[P^{\otimes 3} (h_1 + h_2 + h_3) P^{\otimes 3} \gamma_N^{(3)} \right] + \mathrm{Tr} \left[Q h Q \gamma_N^{(1)} \right]\end{aligned}$$

using that h commutes with P and Q , $PQ = QP = 0$ and the fact that h is a positive operator.

For the two-body term we write

$$\begin{aligned}\mathrm{Tr} \left[W_2 \gamma_N^{(2)} \right] &= \mathrm{Tr} \left[P^{\otimes 3} W_2(1, 2) P^{\otimes 3} \gamma_N^{(3)} \right] + \mathrm{Tr} \left[\Pi_3 W_2(1, 2) \Pi_3 \gamma_N^{(3)} \right] \\ &\quad + \mathrm{Tr} \left[(P^{\otimes 3} W_2(1, 2) \Pi_3 + \Pi_3 W_2(1, 2) P^{\otimes 3}) \gamma_N^{(3)} \right].\end{aligned}$$

Next, since

$$\Pi_3 = P^{\otimes 2} \otimes Q + \Pi_2 \otimes P + \Pi_2 \otimes Q, \quad (3.14)$$

and $W_2(1, 2)$ only acts on the first two variables, this simplifies into

$$\begin{aligned}\mathrm{Tr} \left[W_2 \gamma_N^{(2)} \right] &= \mathrm{Tr} \left[P^{\otimes 3} W_2(1, 2) P^{\otimes 3} \gamma_N^{(3)} \right] + \mathrm{Tr} \left[\Pi_3 W_2(1, 2) \Pi_3 \gamma_N^{(3)} \right] \\ &\quad + \mathrm{Tr} \left[(P^{\otimes 3} W_2(1, 2) \Pi_2 \otimes P + \Pi_2 \otimes P W_2(1, 2) P^{\otimes 3}) \gamma_N^{(3)} \right] \\ &\geq \mathrm{Tr} \left[P^{\otimes 3} W_2(1, 2) P^{\otimes 3} \gamma_N^{(3)} \right] + \mathrm{Tr} \left[\Pi_3 W_2(1, 2) \Pi_3 \gamma_N^{(3)} \right] \\ &\quad - \delta_1 \mathrm{Tr} \left[\Pi_2 \otimes P |W_2(1, 2)| \Pi_2 \otimes P \gamma_N^{(3)} \right] - \delta_1^{-1} \mathrm{Tr} \left[P^{\otimes 3} |W_2(1, 2)| P^{\otimes 3} \gamma_N^{(3)} \right]\end{aligned}$$

where we use (3.12) to obtain the lower bound. Then, using (3.14) and (3.12) again for the second term of the right-hand side, as well as $P, Q \leq \mathbb{1}$, we get

$$\begin{aligned} \operatorname{Tr} \left[W_2 \gamma_N^{(2)} \right] &\geq \operatorname{Tr} \left[P^{\otimes 3} W_2(1, 2) P^{\otimes 3} \gamma_N^{(3)} \right] - \delta_1^{-1} \operatorname{Tr} \left[P^{\otimes 2} |W_2| P^{\otimes 2} \gamma_N^{(2)} \right] \\ &\quad - (3 + \delta_1) \operatorname{Tr} \left[\Pi_2 |W_2| \Pi_2 \gamma_N^{(2)} \right] - 2 \operatorname{Tr} \left[P^{\otimes 2} \otimes Q |W_2(1, 2)| P^{\otimes 2} \otimes Q \gamma_N^{(3)} \right]. \end{aligned}$$

Finally, the three-body term is dealt with similarly:

$$\begin{aligned} \operatorname{Tr} \left[W_3 \gamma_N^{(3)} \right] &= \operatorname{Tr} \left[(P^{\otimes 3} + \Pi_3) W_3 (P^{\otimes 3} + \Pi_3) \gamma_N^{(3)} \right] \\ &\geq \operatorname{Tr} \left[P^{\otimes 3} W_3 P^{\otimes 3} \gamma_N^{(3)} \right] - (1 + \delta_2) \operatorname{Tr} \left[\Pi_3 |W_3| \Pi_3 \gamma_N^{(3)} \right] - \delta_2^{-1} \operatorname{Tr} \left[P^{\otimes 3} |W_3| P^{\otimes 3} \gamma_N^{(3)} \right] \end{aligned}$$

using (3.12) again. All in all, using also the symmetry of $\gamma_N^{(3)}$, we obtain (3.13).

Step 3. We next estimate the remainder terms in (3.13). First we note that

$$\begin{aligned} \operatorname{Tr} \left[h Q \gamma_N^{(1)} Q \right] &\geq \frac{\Lambda}{2} \operatorname{Tr} \left[Q \gamma_N^{(1)} Q \right] + \frac{\sqrt{\Lambda}}{2} \operatorname{Tr} \left[\sqrt{h} Q \gamma_N^{(1)} Q \right] \\ &\geq \frac{\Lambda}{4} \operatorname{Tr} \left[Q \gamma_N^{(1)} Q \right] + \frac{\Lambda}{20} \operatorname{Tr} \left[\Pi_3 \gamma_N^{(3)} \Pi_3 \right] + \frac{\sqrt{\Lambda}}{4} \operatorname{Tr} \left[\sqrt{h_1} \Pi_2 \gamma_N^{(2)} \Pi_2 \right]. \end{aligned} \quad (3.15)$$

The first inequality is just the definition of Q , and to see the second one we first write

$$\begin{aligned} 2 \operatorname{Tr} \left[\sqrt{h} Q \gamma_N^{(1)} Q \right] &= \operatorname{Tr} \left[\sqrt{h_1} Q \otimes \mathbb{1} \gamma_N^{(2)} \right] + \operatorname{Tr} \left[\sqrt{h_2} \mathbb{1} \otimes Q \gamma_N^{(2)} \right] \\ &= \operatorname{Tr} \left[\sqrt{h_1} (Q \otimes P + Q \otimes Q) \gamma_N^{(2)} \right] + \operatorname{Tr} \left[\sqrt{h_2} (P \otimes Q + Q \otimes Q) \gamma_N^{(2)} \right] \\ &= \operatorname{Tr} \left[\sqrt{h_1} (Q \otimes P + Q \otimes Q) \gamma_N^{(2)} (P^{\otimes 2} + \Pi_2) \right] \\ &\quad + \operatorname{Tr} \left[\sqrt{h_2} (P \otimes Q + Q \otimes Q) \gamma_N^{(2)} (P^{\otimes 2} + \Pi_2) \right] \\ &= \operatorname{Tr} \left[\sqrt{h_1} \Pi_2 \gamma_N^{(2)} \Pi_2 \right] + \operatorname{Tr} \left[\sqrt{h_2} Q \otimes Q \gamma_N^{(2)} Q \otimes Q \right] \\ &\quad + \operatorname{Tr} \left[\left(\sqrt{h_2} - \sqrt{h_1} \right) P \otimes Q \gamma_N^{(2)} P \otimes Q \right] \\ &\geq \operatorname{Tr} \left[\sqrt{h_1} \Pi_2 \gamma_N^{(2)} \Pi_2 \right], \end{aligned}$$

where we use repeatedly the cyclicity of the trace and the fact that \sqrt{h} commutes with P and Q , along with the fact that $PQ = QP = 0$ and

$$\Pi_2 = \mathbb{1}^{\otimes 2} - P^{\otimes 2} = Q \otimes Q + P \otimes Q + Q \otimes P.$$

In the last step we also use that as operators

$$\sqrt{h_2} P \otimes Q \geq \sqrt{\Lambda} P \otimes Q \geq \sqrt{h_1} P \otimes Q$$

by definition of the projectors P and Q . This gives the third term in the right-hand side of (3.15). The second one arises from similar considerations:

$$\begin{aligned} \operatorname{Tr} [\Pi_3 \gamma_N^{(3)}] &= \operatorname{Tr} [\Pi_2 \otimes Q \gamma_N^{(3)}] + \operatorname{Tr} [P^{\otimes 2} \otimes Q \gamma_N^{(3)}] + \operatorname{Tr} [\Pi_2 \otimes P \gamma_N^{(3)}] \\ &\leq 2 \operatorname{Tr} [Q \gamma_N^{(1)}] + \operatorname{Tr} [\Pi_2 \gamma_N^{(2)}] \\ &= 2 \operatorname{Tr} [Q \gamma_N^{(1)}] + \operatorname{Tr} [(P \otimes Q + Q \otimes P + Q \otimes Q) \gamma_N^{(2)}] \\ &\leq 5 \operatorname{Tr} [Q \gamma_N^{(1)}]. \end{aligned}$$

Next, using (2.6), we have

$$\operatorname{Tr} [\Pi_2 |W_2| \Pi_2 \gamma_N^{(2)}] \leq \frac{C}{R} \operatorname{Tr} [p_1 | \Pi_2 \gamma_N^{(2)} \Pi_2] \leq \frac{C}{R} \operatorname{Tr} [\sqrt{h_1} \Pi_2 \gamma_N^{(2)} \Pi_2] \quad (3.16)$$

by operator monotonicity of the square-root, and by (2.1)

$$\operatorname{Tr} [\Pi_3 |W_3| \Pi_3 \gamma_N^{(3)}] \leq \frac{C}{R^2} \operatorname{Tr} [\Pi_3 \gamma_N^{(3)} \Pi_3]. \quad (3.17)$$

Moreover, using (2.6) again we get

$$\operatorname{Tr} [P^{\otimes 2} \otimes Q |W_2(1, 2)| P^{\otimes 2} \otimes Q \gamma_N^{(3)}] \leq \frac{C\sqrt{\Lambda}}{R} \operatorname{Tr} [P^{\otimes 2} \otimes Q \gamma_N^{(3)}] \leq \frac{C\sqrt{\Lambda}}{R} \operatorname{Tr} [Q \gamma_N^{(1)}]$$

so that, combining with (3.15), choosing for some small fixed $c_1, c_2 > 0$

$$\delta_1 = c_1 \sqrt{\Lambda} R, \quad \delta_2 = c_2 \Lambda R^2$$

and Λ large enough (i.e. $\Lambda R^2 > c$ for c large enough), we get

$$\begin{aligned} \operatorname{Tr} [h Q \gamma_N^{(1)} Q] - |\beta| (3 + \delta_1) \operatorname{Tr} [\Pi_2 |W_2| \Pi_2 \gamma_N^{(2)}] - 2|\beta| \operatorname{Tr} [P^{\otimes 2} \otimes Q |W_2(1, 2)| P^{\otimes 2} \otimes Q \gamma_N^{(3)}] \\ - \beta^2 (1 + \delta_2) \operatorname{Tr} [\Pi_3 |W_3| \Pi_3 \gamma_N^{(3)}] \geq C \operatorname{Tr} [h Q \gamma_N^{(1)} Q] \end{aligned}$$

for some fixed constant $C > 0$. Then, inserting in (3.13), we deduce

$$\begin{aligned} \operatorname{Tr} [\tilde{H}_3^R \gamma_N^{(3)}] &\geq \operatorname{Tr} [\tilde{H}_3^R P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3}] + C \operatorname{Tr} [h Q \gamma_N^{(1)} Q] \\ &\quad - \frac{C}{\sqrt{\Lambda} R} \operatorname{Tr} [P^{\otimes 2} |W_2| P^{\otimes 2} \gamma_N^{(2)}] \\ &\quad - \frac{C}{\Lambda R^2} \operatorname{Tr} [P^{\otimes 3} |W_3| P^{\otimes 3} \gamma_N^{(3)}]. \end{aligned} \quad (3.18)$$

But, using (2.7), (2.16), and $\operatorname{Tr} \gamma_N^{(k)} = 1$,

$$\begin{aligned} \operatorname{Tr} [P^{\otimes 2} |W_2| P^{\otimes 2} \gamma_N^{(2)}] &\leq C_\varepsilon R^{-\varepsilon} \operatorname{Tr} [P^{\otimes 2} (p_1^2 + 1) P^{\otimes 2} \gamma_N^{(2)}] \\ &\leq C_\varepsilon R^{-\varepsilon} \operatorname{Tr} [(p_1^2 + 1) \gamma_N^{(1)}] \leq C_\varepsilon R^{-\varepsilon}, \end{aligned}$$

whereas, using (2.10) and (2.16) again

$$\begin{aligned} \mathrm{Tr} \left[P^{\otimes 3} |W_3| P^{\otimes 3} \gamma_N^{(3)} \right] &\leq C \mathrm{Tr} \left[P^{\otimes 3} (p_1^2 + 1) P^{\otimes 3} \gamma_N^{(3)} \right] \\ &\leq C \mathrm{Tr} \left[(p_1^2 + 1) \gamma_N^{(1)} \right] \leq C, \end{aligned}$$

which completes the proof. \square

3.3. Energy bounds. In this subsection we prove the energy bounds establishing (1.19). The upper bound is obtained as usual by testing against a factorized trial state. Namely, taking $\Psi_N = (u_R^{\mathrm{af}})^{\otimes N}$ in (1.11) with u_R^{af} a normalized minimizer of $\mathcal{E}_R^{\mathrm{af}}$, and using (1.14), (1.16), Lemmas 2.2, 2.3, 2.4, and the diamagnetic inequality (A.3), one finds

$$\frac{E^R(N)}{N} \leq \mathcal{E}_R^{\mathrm{af}}[u_R^{\mathrm{af}}] + (\mathcal{E}_R^{\mathrm{af}}[u_R^{\mathrm{af}}] + 1) \left(\frac{C\beta^2}{N} + \frac{C_\varepsilon \beta^2 R^{-\varepsilon}}{N} \right) = E_R^{\mathrm{af}} + o(1) \rightarrow E^{\mathrm{af}}, \quad (3.19)$$

as $R \sim N^{-\eta}$ with $N \rightarrow \infty$, where we also used Proposition A.6. Note that for this upper bound we can allow any rate $0 < \eta < \infty$, and may even take $\beta = \beta(N) \rightarrow \infty$.

For the lower bound, inserting (3.8) in the estimate of Proposition 3.3 we get for any sequence of ground states Ψ_N that

$$\begin{aligned} \frac{1}{N} \langle \Psi_N, H_N^R \Psi_N \rangle &\geq \mathrm{Tr} \left[\tilde{H}_3^R \tilde{\gamma}_N^{(3)} \right] + C\Lambda \mathrm{Tr} [Q\gamma_N^{(1)}] - C \frac{N_\Lambda}{N} \left\| P^{\otimes 3} \tilde{H}_3^R P^{\otimes 3} \right\| \\ &\quad - C \left(\frac{1}{N} + \frac{C_\varepsilon}{\sqrt{\Lambda} R^{1+\varepsilon}} + \frac{1}{\Lambda R^2} \right) \\ &\geq \mathrm{Tr} \left[\tilde{H}_3^R \tilde{\gamma}_N^{(3)} \right] + C\Lambda \mathrm{Tr} [Q\gamma_N^{(1)}] - C \frac{\Lambda^{2+2/s}}{N} (1 + |\log R|) \\ &\quad - C \left(\frac{1}{N} + \frac{C_\varepsilon}{\sqrt{\Lambda} R^{1+\varepsilon}} + \frac{1}{\Lambda R^2} \right). \end{aligned} \quad (3.20)$$

We have here used Estimates (2.8) and (2.10) from Subsection 2.1, along with

$$Pp^2P \leq PhP \leq \Lambda$$

to bound the operator norm of $P^{\otimes 3} \tilde{H}_3^R P^{\otimes 3}$ and Lemma 3.1 to bound N_Λ .

Main term. Since by definition $\tilde{\gamma}_N^{(3)}$ is a superposition of tensorized states we get

$$\mathrm{Tr} \left[\tilde{H}_3^R \tilde{\gamma}_N^{(3)} \right] \geq E_R^{\mathrm{af}} \mathrm{Tr} \left[\tilde{\gamma}_N^{(3)} \right].$$

We then denote

$$\lambda := \mathrm{Tr} \left[P\gamma_N^{(1)} \right] = \sum_{k=0}^N \frac{k}{N} \mathrm{Tr} \left[G_{N,k}^P \right] \quad (3.21)$$

the fraction of P -localized particles. Using the simple estimate

$$\left| \frac{k}{N} \frac{k-1}{N-1} \frac{k-2}{N-2} - \frac{k^3}{N^3} \right| \leq CN^{-1} \text{ for } 0 \leq k \leq N,$$

it follows from (3.5), (3.6), (3.7), and Jensen's inequality that

$$\begin{aligned} \mathrm{Tr} \left[\tilde{\gamma}_N^{(3)} \right] &= \int_{SP\mathfrak{H}} d\mu_N = \sum_{k=3}^N \binom{N}{3}^{-1} \binom{k}{3} \mathrm{Tr} [G_{N,k}^P] \\ &\geq \sum_{k=0}^N \left(\frac{k}{N} \right)^3 \mathrm{Tr} [G_{N,k}^P] - O(N^{-1}) \\ &\geq \lambda^3 - O(N^{-1}). \end{aligned} \quad (3.22)$$

Since on the other hand

$$\mathrm{Tr} \left[Q\gamma_N^{(1)} \right] = 1 - \lambda,$$

we conclude

$$\mathrm{Tr} \left[\tilde{H}_3^R \tilde{\gamma}_N^{(3)} \right] + C\Lambda \mathrm{Tr} [Q\gamma_N^{(1)}] \geq \lambda^3 E_R^{\mathrm{af}} + C\Lambda(1 - \lambda) - CN^{-1} \geq E_R^{\mathrm{af}} - CN^{-1}. \quad (3.23)$$

For the last inequality we bound from below in terms of the infimum with respect to $0 \leq \lambda \leq 1$. Since Λ is a very large number and E_R^{af} is bounded as $R \rightarrow 0$ (see Proposition A.6), the infimum is clearly attained at $\lambda = 1$.

Optimizing the error. We next choose Λ to minimize the error in (3.20). We assume that R behaves at worst as

$$R \sim N^{-\eta}.$$

Changing a little bit η if necessary we may ignore the $|\log R|$ and R^ε factors, and we thus have to minimize

$$\frac{1}{\sqrt{\Lambda}R} + \frac{1}{\Lambda R^2} + \frac{\Lambda^{2+2/s}}{N}.$$

We pick

$$\Lambda = N^{\frac{2}{5}(1+\eta)(1+4/5s)^{-1}}$$

to equate the first and the last term and get

$$\frac{1}{\sqrt{\Lambda}R} + \frac{1}{\Lambda R^2} + \frac{\Lambda^{2(1+1/s)}}{N} = O(N^{\frac{4}{5}(1+\eta)(1+1/s)(1+4/5s)^{-1}-1}) + O(N^{\frac{8}{5}(1+\eta)(1+1/s)(1+4/5s)^{-1}-2}),$$

and this is small provided

$$\eta < \eta_0 := \frac{1}{4} \left(1 + \frac{1}{s} \right)^{-1}.$$

Since this is the main error term we conclude that the lower bound corresponding to (1.19) holds provided $R \sim N^{-\eta}$ with $\eta < \eta_0$, as stated in the theorem. The limit $E_R^{\mathrm{af}} \rightarrow E^{\mathrm{af}}$ is dealt with in Appendix A.

3.4. Convergence of states. Given the previous constructions and energy estimates, the proof of (1.20) follows almost exactly [26, Section 4.3] and is thus only sketched.

Modulo extraction of subsequences we have

$$\gamma_N^{(k)} \rightharpoonup_* \gamma^{(k)}$$

weakly-* in the trace-class as $N \rightarrow \infty$. From Proposition 2.6 we know that $(-\Delta + V)\gamma_N^{(1)}$ is uniformly bounded in trace-class. Under our assumptions, $(-\Delta + V)^{-1}$ is compact and we may thus, modulo a further extraction, assume that

$$\gamma_N^{(1)} \xrightarrow{N \rightarrow \infty} \gamma^{(1)}$$

strongly in trace-class norm. Then, by [25, Corollary 2.4], we also have

$$\gamma_N^{(k)} \xrightarrow{N \rightarrow \infty} \gamma^{(k)}$$

strongly for all $k \geq 0$.

Next we claim that the measure μ_N defined in Lemma 3.2 converges (modulo extraction) to a limit probability measure $\mu \in \mathcal{P}(S\mathfrak{H})$ on the unit sphere of $\mathfrak{H} = L^2(\mathbb{R}^2)$ and that

$$\gamma^{(k)} = \int_{u \in S\mathfrak{H}} |u^{\otimes k}\rangle \langle u^{\otimes k}| d\mu(u) \quad \text{for all } k \geq 0. \quad (3.24)$$

To see this, we first apply (3.8), with the above choice of Λ . We obtain

$$\text{Tr} \left| P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3} - \int_{SP\mathfrak{H}} |u^{\otimes 3}\rangle \langle u^{\otimes 3}| d\mu_N(u) \right| \rightarrow 0. \quad (3.25)$$

On the other hand, combining (3.23) with the energy upper bound (3.19) we get

$$\lambda \rightarrow 1$$

where λ is the fraction of P -localized particles defined in (3.21). Using Jensen's inequality as in (3.22) we deduce that

$$\mu_N(SP\mathfrak{H}) = \text{Tr} \left[P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3} \right] \rightarrow 1. \quad (3.26)$$

Combining with (3.25) yields

$$\text{Tr} \left| \gamma_N^{(3)} - \int_{SP\mathfrak{H}} |u^{\otimes 3}\rangle \langle u^{\otimes 3}| d\mu_N(u) \right| \rightarrow 0. \quad (3.27)$$

Testing this with a sequence of finite rank orthogonal projectors

$$P_K \xrightarrow{K \rightarrow \infty} \mathbb{1}$$

and using the strong convergence of $\gamma_N^{(3)}$ gives

$$\lim_{K \rightarrow \infty} \lim_{N \rightarrow \infty} \mu_N(P_K \mathfrak{H}) = 1,$$

and we obtain the existence of a limit measure μ supported on the unit ball of \mathfrak{H} by a tightness argument. Then (3.24) for $k = 3$ follows from (3.27). Since $\gamma_N^{(3)}$ converges strongly, the limit has trace 1 and μ must be supported on the unit sphere. Obtaining (3.24) for larger k is a general argument based on (3.26). We refer to [26, Section 4.3] for details.

There only remains to prove that μ is supported on \mathcal{M}^{af} . But it follows easily from combining previously obtained energy bounds that

$$\int_{SP\mathfrak{H}} \left| E_R^{\text{af}} - \mathcal{E}_R^{\text{af}}[u] \right| d\mu_N(u) \xrightarrow{N \rightarrow \infty} 0.$$

Using in addition the results of Appendix A, in particular Proposition A.6, we obtain for a large but fixed constant $C > 0$

$$\int_{\mathcal{E}^{\text{af}}[u] \leq C} \left| E^{\text{af}} - \mathcal{E}^{\text{af}}[u] \right| d\mu_N(u) \xrightarrow{N \rightarrow \infty} 0$$

and

$$\int_{\mathcal{E}^{\text{af}}[u] \geq C} d\mu_N(u) \xrightarrow{N \rightarrow \infty} 0.$$

Then clearly μ must be supported on \mathcal{M}^{af} , which concludes the proof. \square

APPENDIX A. PROPERTIES OF THE AVERAGE-FIELD FUNCTIONAL

In this appendix we establish some of the fundamental properties of the functional (1.10) and its limit $R \rightarrow 0$.

For $\beta \in \mathbb{R}$ and $V : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ we define the average-field energy functional

$$\mathcal{E}^{\text{af}}[u] := \int_{\mathbb{R}^2} \left(\left| (\nabla + i\beta \mathbf{A}[|u|^2]) u \right|^2 + V|u|^2 \right), \quad (\text{A.1})$$

with the self-generated magnetic potential

$$\mathbf{A}[\rho] := \nabla^\perp w_0 * \rho = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \rho(y) dy, \quad \text{curl } \mathbf{A}[\rho] = 2\pi\rho.$$

The functional is certainly well-defined for $u \in C_c^\infty(\mathbb{R}^2)$, but we should ask what its natural domain is. We then have to make a meaning of $\mathcal{E}^{\text{af}}[u]$ for general $u \in L^2(\mathbb{R}^2)$ and the problem is that it is not certain that $\mathbf{A}[|u|^2] \in L_{\text{loc}}^2$ even though $u \in L^2$ (see [12] for an example³), so the product $\mathbf{A}[|u|^2]u$ may not be well-defined as a distribution (while ∇u certainly is). One way around this is to reconsider the form of the functional when acting on regular enough functions such that we can write $u = |u|e^{i\varphi}$ where φ is real. Then

$$\left| (\nabla + i\beta \mathbf{A}[|u|^2])u \right|^2 = \left| \nabla|u| + i|u|(\nabla\varphi + \beta \mathbf{A}[|u|^2]) \right|^2 = |\nabla|u||^2 + \left| |u|\nabla\varphi + \beta \mathbf{A}[|u|^2]|u| \right|^2,$$

where also $\nabla\varphi = |u|^{-2} \Im \bar{u} \nabla u$ and $\nabla|u| = |u|^{-1} \Re \bar{u} \nabla u$. Hence, an alternative definition is given by

$$\mathcal{E}^{\text{af}}[u] := \int_{\mathbb{R}^2} \left(|\nabla|u||^2 + \left| \Im \frac{\bar{u}}{|u|} \nabla u + \beta \mathbf{A}[|u|^2]|u| \right|^2 + V|u|^2 \right), \quad (\text{A.2})$$

and the advantage of this formulation is that it makes clear that we actually demand $|u| \in H^1(\mathbb{R}^2)$ in order for $\mathcal{E}^{\text{af}}[u] < \infty$. We can then use the following lemma to see that in fact $\mathbf{A}[|u|^2]u \in L^2(\mathbb{R}^2)$, and hence also $\nabla u \in L^2(\mathbb{R}^2)$. (And conversely this also shows that if $\mathbf{A}[|u|^2]u \notin L^2(\mathbb{R}^2)$ then we have no chance of making sense out of $\mathcal{E}^{\text{af}}[u]$.)

Lemma A.1 (Bound on the magnetic term).

We have for any $u \in L^2(\mathbb{R}^2)$ that

$$\int_{\mathbb{R}^2} \left| \mathbf{A}[|u|^2] \right|^2 |u|^2 \leq \frac{3}{2} \|u\|_{L^2(\mathbb{R}^2)}^4 \int_{\mathbb{R}^2} |\nabla|u||^2.$$

³Note that by Young's inequality we have for any $u \in L^2(\mathbb{R}^2)$ that $\mathbf{A}[|u|^2] \in L^p(\mathbb{R}^2) + \varepsilon L^\infty(\mathbb{R}^2)$ for $p \in [1, 2)$. Also compare to the singular magnetic fields considered in [12, 36].

Proof. This follows from symmetry and from the three-body Hardy inequality of Lemma 2.5:

$$\begin{aligned} \int_{\mathbb{R}^2} |\mathbf{A}[|u|^2](x)|^2 |u(x)|^2 dx &= \iiint_{\mathbb{R}^6} \frac{x-y}{|x-y|^2} \cdot \frac{x-z}{|x-z|^2} |u(x)|^2 |u(y)|^2 |u(z)|^2 dx dy dz \\ &= \frac{1}{6} \int_{\mathbb{R}^6} \frac{1}{\mathcal{R}(X)^2} |u|^{\otimes 3}|^2 dX \leq \frac{1}{2} \int_{\mathbb{R}^6} |\nabla_X |u|^{\otimes 3}|^2 dX = \frac{3}{2} \int_{\mathbb{R}^2} |\nabla |u||^2 dx \left(\int_{\mathbb{R}^2} |u|^2 dx \right)^2. \end{aligned}$$

□

We can therefore define the domain of \mathcal{E}^{af} to be (and otherwise let $\mathcal{E}^{\text{af}}[u] := +\infty$)

$$\mathcal{D}^{\text{af}} := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V|u|^2 < \infty \right\},$$

and we find using Cauchy-Schwarz, Lemma A.1, and $|\nabla |u|| \leq |\nabla u|$ that for $u \in \mathcal{D}^{\text{af}}$

$$0 \leq \mathcal{E}^{\text{af}}[u] \leq 2\|\nabla u\|^2 + 2\beta^2 \|\mathbf{A}[|u|^2]u\|^2 + \int V|u|^2 \leq (2 + 3\beta^2 \|u\|^4) \|\nabla u\|^2 + \int V|u|^2 < \infty.$$

The ground-state energy of the average-field functional is then given by

$$E^{\text{af}} := \inf \left\{ \mathcal{E}^{\text{af}}[u] : u \in \mathcal{D}^{\text{af}}, \int_{\mathbb{R}^2} |u|^2 = 1 \right\}.$$

For convenience we also make the assumption on V that $V(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$ and that $C_c^\infty(\mathbb{R}^2) \subseteq \mathcal{D}^{\text{af}}$ is a form core for $\|u\|_{L_V^2}^2 := \int_{\mathbb{R}^2} V|u|^2$, with $-\Delta + V$ essentially self-adjoint and with purely discrete spectrum (see, e.g., [46, Theorem XIII.67]). This is then also a core for \mathcal{E}^{af} :

Proposition A.2 (Density of regular functions in the form domain).

$C_c^\infty(\mathbb{R}^2)$ is dense in \mathcal{D}^{af} w.r.t. \mathcal{E}^{af} , namely for any $u \in \mathcal{D}^{\text{af}}$ there exists a sequence $(u_n)_{n \rightarrow \infty} \subset C_c^\infty(\mathbb{R}^2)$ such that

$$\|u - u_n\|_{H^1} \rightarrow 0 \text{ and } \mathcal{E}^{\text{af}}[u_n] \rightarrow \mathcal{E}^{\text{af}}[u] \text{ as } n \rightarrow \infty.$$

Proof. Take $u \in \mathcal{D}^{\text{af}}$, then $\|\nabla u\|_{L^2} < \infty$ and hence also $\|u\|_{L^p} < \infty$ for any $p \in [2, \infty)$ by Sobolev embedding. We use that $C_c^\infty(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, so there exists a sequence $(u_n)_{n \rightarrow \infty} \subset C_c^\infty$ s.t. $\|u - u_n\|_{H^1} \rightarrow 0$. Also,

$$\begin{aligned} & \left| \|(\nabla + i\beta \mathbf{A}[|u|^2])u\|_2 - \|(\nabla + i\beta \mathbf{A}[|u_n|^2])u_n\|_2 \right| \\ & \leq \|(\nabla + i\beta \mathbf{A}[|u|^2])u - (\nabla + i\beta \mathbf{A}[|u_n|^2])u_n\|_2 \\ & \leq \|\nabla(u - u_n)\|_2 + |\beta| \|(\mathbf{A}[|u|^2] - \mathbf{A}[|u_n|^2])u + \mathbf{A}[|u_n|^2](u - u_n)\|_2 \\ & \leq \|u - u_n\|_{H^1} + |\beta| \|\mathbf{A}[|u|^2 - |u_n|^2]u\|_2 + |\beta| \|\mathbf{A}[|u_n|^2](u - u_n)\|_2, \end{aligned}$$

where by Hölder's and generalized Young's inequalities

$$\begin{aligned} \|\mathbf{A}[|u|^2 - |u_n|^2]u\|_2 &\leq \|\mathbf{A}[|u|^2 - |u_n|^2]\|_4 \|u\|_4 \leq C \| |u|^2 - |u_n|^2 \|_{4/3} \|\nabla w_0\|_{2,w} \|u\|_4 \\ &\leq C' \|u - u_n\|_{8/3} \leq C'' \|u - u_n\|_{H^1} \rightarrow 0, \end{aligned}$$

and similarly

$$\|\mathbf{A}[|u_n|^2](u - u_n)\|_2 \leq C \|u - u_n\|_{H^1} \rightarrow 0,$$

as $n \rightarrow \infty$.

We also have continuity for $\|u\|_{L_V^2}$ here since we assumed that $C_c^\infty(\mathbb{R}^2)$ is a form core. □

Lemma A.3 (Basic magnetic inequalities).

We have for $u \in \mathcal{D}^{\text{af}}$ that (diamagnetic inequality)

$$\int_{\mathbb{R}^2} |(\nabla + i\beta \mathbf{A}[|u|^2])u|^2 \geq \int_{\mathbb{R}^2} |\nabla u|^2, \quad (\text{A.3})$$

and

$$\int_{\mathbb{R}^2} |(\nabla + i\beta \mathbf{A}[|u|^2])u|^2 \geq 2\pi|\beta| \int_{\mathbb{R}^2} |u|^4. \quad (\text{A.4})$$

Proof. By density we can w.l.o.g. assume $u \in C_c^\infty(\mathbb{R}^2)$. We then have $\mathbf{A}[|u|^2] \in C^\infty(\mathbb{R}^2) \subseteq L_{\text{loc}}^2(\mathbb{R}^2)$ and hence the first inequality follows by the usual diamagnetic inequality (see e.g. Theorem 2.1.1 in [14]). Furthermore, by e.g. Lemma 1.4.1 in [14],

$$\int_{\mathbb{R}^2} |(\nabla + i\beta \mathbf{A}[|u|^2])u|^2 \geq \pm \int_{\mathbb{R}^2} \text{curl}(\beta \mathbf{A}[|u|^2]) |u|^2,$$

which proves the second inequality since $\text{curl} \mathbf{A}[|u|^2] = 2\pi|u|^2$. Instead of using density we could also have used the formulation (A.2) or the fact that $u \in H^1 \Rightarrow \mathbf{A}[|u|^2] \in L^p$, $p \in (2, \infty)$ by generalized Young. \square

Proposition A.4 (Existence of minimizers).

For any value of $\beta \in \mathbb{R}$ there exists $u^{\text{af}} \in \mathcal{D}^{\text{af}}$ with $\int_{\mathbb{R}^2} |u^{\text{af}}|^2 = 1$ and $\mathcal{E}^{\text{af}}[u^{\text{af}}] = E^{\text{af}}$.

Proof. First note that for $u \in \mathcal{D}^{\text{af}}$, by Lemma A.1 and Lemma A.3,

$$\begin{aligned} \|\nabla u\|_2 &= \|\nabla u + i\beta \mathbf{A}[|u|^2]u - i\beta \mathbf{A}[|u|^2]u\|_2 \leq \mathcal{E}^{\text{af}}[u]^{1/2} + |\beta| \|\mathbf{A}[|u|^2]u\|_2 \\ &\leq \mathcal{E}^{\text{af}}[u]^{1/2} + |\beta| \sqrt{\frac{3}{2}} \|u\|_2^2 \|\nabla u\|_2 \leq \left(1 + |\beta| \sqrt{\frac{3}{2}} \|u\|_2^2\right) \mathcal{E}^{\text{af}}[u]^{1/2}. \end{aligned}$$

Now take a minimizing sequence

$$(u_n)_{n \rightarrow \infty} \subset \mathcal{D}^{\text{af}}, \quad \|u_n\|_2 = 1, \quad \lim_{n \rightarrow \infty} \mathcal{E}^{\text{af}}[u_n] = E^{\text{af}}.$$

Then clearly (u_n) is uniformly bounded in both $L^2(\mathbb{R}^2)$, L_V^2 , and $H^1(\mathbb{R}^2)$ (and hence in $L^p(\mathbb{R}^2)$, $p \in [2, \infty)$), and therefore by the Banach-Alaoglu theorem there exists $u^{\text{af}} \in \mathcal{D}^{\text{af}}$ and a weakly convergent subsequence (still denoted u_n) such that

$$u_n \rightharpoonup u^{\text{af}} \text{ in } L^2(\mathbb{R}^2) \cap L_V^2 \cap L^p(\mathbb{R}^2), \quad \nabla u_n \rightharpoonup \nabla u^{\text{af}} \text{ in } L^2(\mathbb{R}^2).$$

Moreover, since $(-\Delta + V + 1)^{-1/2}$ is compact we have that

$$u_n = (-\Delta + V + 1)^{-1/2} (-\Delta + V + 1)^{1/2} u_n$$

is actually strongly convergent (again extracting a subsequence), hence

$$u_n \rightarrow u^{\text{af}} \text{ in } L^2(\mathbb{R}^2).$$

Also, $\mathbf{A}[|u_n|]$ converges pointwise a.e. to $\mathbf{A}[|u|^2]$ by weak convergence of u_n in L^p and, by the trick of Lemma A.1,

$$\|\mathbf{A}[|u_n|^2]u_n\|_2^2 = \frac{1}{6} \int_{\mathbb{R}^6} \mathcal{R}(X)^{-2} ||u_n|^{\otimes 3}|^2 dX \rightarrow \frac{1}{6} \int_{\mathbb{R}^6} \mathcal{R}(X)^{-2} ||u|^{\otimes 3}|^2 dX = \|\mathbf{A}[|u|^2]u\|_2^2$$

by dominated convergence. The functions $\mathbf{A}[|u_n|^2]u_n$ are therefore even strongly converging to $\mathbf{A}[|u|^2]u$ in $L^2(\mathbb{R}^2)$ by dominated convergence. It then follows that

$$\begin{aligned} \|(\nabla + i\beta\mathbf{A}[|u|^2])u\|_2 &= \sup_{\|v\|=1} |\langle \nabla u + i\beta\mathbf{A}[|u|^2]u, v \rangle| \\ &= \sup_{\|v\|=1} \lim_{n \rightarrow \infty} |\langle \nabla u_n + i\beta\mathbf{A}[|u_n|^2]u_n, v \rangle| \\ &\leq \liminf_{n \rightarrow \infty} \sup_{\|v\|=1} |\langle \nabla u_n + i\beta\mathbf{A}[|u_n|^2]u_n, v \rangle| \\ &= \liminf_{n \rightarrow \infty} \|(\nabla + i\beta\mathbf{A}[|u_n|^2])u_n\|_2, \end{aligned}$$

and since $\|\cdot\|_{L_V^2}$ is also weakly lower semicontinuous (see, e.g., [44, Supplement to IV.5]), we have $\liminf_{n \rightarrow \infty} \mathcal{E}^{\text{af}}[u_n] \geq \mathcal{E}^{\text{af}}[u^{\text{af}}]$. Thus, with $\|u^{\text{af}}\| = \lim_{n \rightarrow \infty} \|u_n\| = 1$, we also have $\mathcal{E}^{\text{af}}[u^{\text{af}}] = E^{\text{af}}$. \square

Proposition A.5 (Convergence to bosons).

Let E_0 resp. u_0 denote the ground-state eigenvalue resp. normalized eigenfunction of the non-magnetic Schrödinger operator $H_1 = -\Delta + V$, with $V \in L_{\text{loc}}^\infty$. We have

$$E_{(\beta)}^{\text{af}} \xrightarrow{\beta \rightarrow 0} E_0,$$

and that given an arbitrary sequence (u_β) of minimizers for $\mathcal{E}_{(\beta)}^{\text{af}}$

$$u_\beta \xrightarrow{\beta \rightarrow 0} u_0 \text{ in } L^2(\mathbb{R}^2)$$

up to a subsequence and a constant phase.

Proof. Note that under our conditions for V , $u_0 \in \mathcal{D}^{\text{af}}$ is the unique minimizer of $\mathcal{E}_0 = \mathcal{E}_{(\beta=0)}^{\text{af}}$ and can be taken positive (see, e.g., [29, Theorem 11.8]). By the diamagnetic inequality (A.3), and by taking the trial state $u_0 = |u_0|$ in $\mathcal{E}_{(\beta \neq 0)}^{\text{af}}$, we find

$$E_0 \leq E_{(\beta)}^{\text{af}} \leq \mathcal{E}_{(\beta)}^{\text{af}}[u_0] = \mathcal{E}_0[u_0] + \beta^2 \|\mathbf{A}[|u_0|^2]u_0\|_2^2 \leq (1 + C\beta^2)E_0$$

(where we also used Lemma A.1), and hence $E_{(\beta)}^{\text{af}} \rightarrow E_0$ as $\beta \rightarrow 0$. Now consider a sequence $(u_\beta) \subset \mathcal{D}^{\text{af}}$ of minimizers as $\beta \rightarrow 0$ with $\mathcal{E}^{\text{af}}[u_\beta] \rightarrow E_0$, $\|u_\beta\| = 1$. Then, because of uniform boundedness and as in the proof of Proposition A.4, we have after taking a subsequence that $u_\beta \rightarrow u$ for some $u \in \mathcal{D}^{\text{af}}$, $\|u\| = 1$, and also

$$\begin{aligned} \|\nabla u\| &= \sup_{\|v\|=1} |\langle \nabla u, v \rangle| \\ &= \sup_{\|v\|=1} \lim_{\beta \rightarrow 0} |\langle \nabla u_\beta + i\beta\mathbf{A}[|u_\beta|^2]u_\beta, v \rangle| \\ &\leq \liminf_{\beta \rightarrow 0} \|\nabla u_\beta + i\beta\mathbf{A}[|u_\beta|^2]u_\beta\|, \end{aligned}$$

so

$$E_0 \leq \mathcal{E}_0[u] \leq \liminf_{\beta \rightarrow 0} \mathcal{E}_{(\beta)}^{\text{af}}[u_\beta].$$

It follows that $\mathcal{E}_0[u] = E_0$ and hence $u = u_0$ up to a constant phase. \square

From the bound (A.4) we observe that the self-generated magnetic interaction is stronger than a contact interaction of strength $2\pi|\beta|$ (despite the fact that we already removed a singular repulsive interaction in the initial regularization step for extended anyons). Hence we have not only $E^{\text{af}} \geq E_0$ by the diamagnetic inequality, but also

$$E^{\text{af}} \geq \min_{\rho \geq 0, \int_{\mathbb{R}^2} \rho = 1} \int_{\mathbb{R}^2} (2\pi|\beta|\rho^2 + V\rho), \quad (\text{A.5})$$

which can be computed for given V by straightforward optimization.

Let us now consider the corresponding situation for the regularized functional (extended anyons)

$$\mathcal{E}_R^{\text{af}}[u] := \int_{\mathbb{R}^2} \left(|(\nabla + i\beta \mathbf{A}^R[|u|^2])u|^2 + V|u|^2 \right), \quad \mathbf{A}^R[\rho] := \nabla^\perp w_R * \rho, \quad R > 0.$$

Since $\nabla w_R \in L^\infty(\mathbb{R}^2)$ we have $\mathbf{A}^R[|u|^2] \in L^\infty(\mathbb{R}^2)$ with

$$\|\mathbf{A}^R[|u|^2]\|_\infty \leq \frac{C}{R} \|u\|_2^2$$

and instead of Lemma A.1 we have

$$\|\mathbf{A}^R[|u|^2]u\|_2 \leq C \|u\|_2^2 \|u\|_{H^1}$$

using Lemma 2.4. Hence the natural domain is again \mathcal{D}^{af} and all properties established above for \mathcal{E}^{af} are also found to be valid for $\mathcal{E}_R^{\text{af}}$ (except (A.4) and (A.5) which now have regularized versions). Denoting

$$E_R^{\text{af}} := \min\{\mathcal{E}_R^{\text{af}}[u] : u \in \mathcal{D}^{\text{af}}, \|u\|_2 = 1\},$$

we furthermore have the following relationship:

Proposition A.6 (Convergence to point-like anyons).

The functional $\mathcal{E}_R^{\text{af}}$ converges pointwise to \mathcal{E}^{af} as $R \rightarrow 0$. More precisely, for any $u \in \mathcal{D}^{\text{af}}$

$$\left| \mathcal{E}_R^{\text{af}}[u] - \mathcal{E}^{\text{af}}[u] \right| \leq C_u |\beta| (1 + \beta^4) (1 + \mathcal{E}^{\text{af}}[u])^{3/2} R, \quad (\text{A.6})$$

where C_u depends only on $\|u\|_2$. Hence,

$$E_R^{\text{af}} \xrightarrow{R \rightarrow 0} E^{\text{af}},$$

and if $(u_R)_{R \rightarrow 0} \subset \mathcal{D}^{\text{af}}$ denotes a sequence of minimizers of $\mathcal{E}_R^{\text{af}}$, then there exists a subsequence $(u_{R'})_{R' \rightarrow 0}$ s.t. $u_{R'} \rightarrow u^{\text{af}}$ as $R' \rightarrow 0$, where u^{af} is some minimizer of \mathcal{E}^{af} .

Proof. We have for any $u \in \mathcal{D}^{\text{af}}$ that

$$\begin{aligned} & \left| \|(\nabla + i\beta \mathbf{A}[|u|^2])u\|_2 - \|(\nabla + i\beta \mathbf{A}^R[|u|^2])u\|_2 \right| \\ & \leq \|(\nabla + i\beta \mathbf{A}[|u|^2])u - (\nabla + i\beta \mathbf{A}^R[|u|^2])u\|_2 = |\beta| \|(\mathbf{A}[|u|^2] - \mathbf{A}^R[|u|^2])u\|_2 \\ & \leq |\beta| \|\mathbf{A}[|u|^2] - \mathbf{A}^R[|u|^2]\|_4 \|u\|_4 = |\beta| \|(\nabla w_0 - \nabla w_R) * |u|^2\|_4 \|u\|_4, \end{aligned}$$

where by Young

$$\|(\nabla w_0 - \nabla w_R) * |u|^2\|_4 \leq \|\nabla w_0 - \nabla w_R\|_1 \| |u|^2 \|_4 \leq \|\nabla w_0\|_{L^1(B(0,R))} \|u\|_8^2 \rightarrow 0,$$

as $R \rightarrow 0$, since $\nabla w_0 \in L^1_{\text{loc}}(\mathbb{R}^2)$. We deduce (A.6) by combining this with previous estimates of this appendix and Sobolev embeddings. It follows that $\mathcal{E}_R^{\text{af}}[u] \rightarrow \mathcal{E}^{\text{af}}[u]$ as $R \rightarrow 0$.

Let $(u_R)_{R \rightarrow 0}$ denote a sequence of minimizers of $\mathcal{E}_R^{\text{af}}$:

$$E_R^{\text{af}} = \mathcal{E}_R^{\text{af}}[u_R], \quad \|u_R\| = 1,$$

and take $u \in \mathcal{D}^{\text{af}}$ an arbitrary minimizer of \mathcal{E}^{af} . Then, since

$$E_R^{\text{af}} \leq \mathcal{E}_R^{\text{af}}[u] \xrightarrow{R \rightarrow 0} \mathcal{E}^{\text{af}}[u] = E^{\text{af}},$$

we have that E_R^{af} is uniformly bounded as $R \rightarrow 0$ and that

$$\limsup_{R \rightarrow 0} E_R^{\text{af}} \leq E^{\text{af}}.$$

Then $\mathcal{E}_R^{\text{af}}[u_R]$, and hence also

$$\mathcal{E}^{\text{af}}[u_R] \leq C \left(\|u_R\|_{H^1}^2 + \|u_R\|_{L^2_V}^2 \right) \leq C'(\mathcal{E}_R^{\text{af}}[u_R] + 1),$$

are uniformly bounded as well. As in the proof of Proposition A.4, there then exists a strongly convergent subsequence $(u_{R'})_{R' \rightarrow 0}$, with $u_{R'} \rightarrow u_0 \in \mathcal{D}^{\text{af}}$. Also, by weak lower semicontinuity $\mathcal{E}^{\text{af}}[u_0] \leq \liminf_{R' \rightarrow 0} \mathcal{E}^{\text{af}}[u_{R'}]$, so that for any $\varepsilon > 0$ and sufficiently small $R' > 0$,

$$E^{\text{af}} \leq \mathcal{E}^{\text{af}}[u_0] \leq \mathcal{E}^{\text{af}}[u_{R'}] + \varepsilon \leq \mathcal{E}_{R'}^{\text{af}}[u_{R'}] + 2\varepsilon = E_{R'}^{\text{af}} + 2\varepsilon,$$

where we also used that the convergence is uniform for our uniformly bounded sequence u_R by the bound (A.6). It follows that $E^{\text{af}} \leq \mathcal{E}^{\text{af}}[u_0] \leq E^{\text{af}} + 3\varepsilon$, and hence u_0 is a minimizer with $\|u_0\| = 1$ and $E^{\text{af}} = \mathcal{E}^{\text{af}}[u_0] = \lim_{R \rightarrow 0} E_R^{\text{af}}$. \square

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